SELF-EQUIVALENces OF (n-1)-CONNECTED 2n-MANIFOLDS

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1. Introduction and statement of main results. All spaces have basepoints, and all maps of spaces are basepoint-preserving. A self-equivalence of a space $X$ is a homotopy class of homotopy equivalences $X \rightarrow X$. Map-composition induces an operation on the set of self-equivalences of $X$, making it into a group, $\mathcal{E}(X)$.

Arkowitz and Curjel [1] and Weishu Shih [7] have obtained certain general results about $\mathcal{E}(X)$ by studying the Postnikov decomposition of $X$. More recently P. Olum [5] presented an explicit computation of $\mathcal{E}(X)$ in the case that $X$ is a pseudo-projective plane.

Our results concern the structure of $\mathcal{E}(X)$ in the case that $X$ is a closed, compact, oriented, $C^\infty$, $(n-1)$-connected $2n$-manifold, $n \geq 2$. We place these restrictions on $X$ throughout the rest of this paper. Our methods are dual to those of [1] and [7] in the sense that we proceed by examining a cell-decomposition of $X$.

A word about notation: $X_n$ is the $n$-skeleton of $X$ in some fixed, minimal CW-decomposition of $X$, $S^X_n$ is its suspension, and $\pi(SX_n, X)$ is the group of homotopy classes of maps $SX_n \rightarrow X$.

**Theorem 1.** There is an exact sequence,

$$
\pi(SX_n, X) \xrightarrow{(Sb)^* + \psi} \pi_{2n}(X) \xrightarrow{\rho} \mathcal{E}(X) \rightarrow \mathcal{E}(X_n),
$$

the homomorphisms of which will be described in §2.

It is easy to show that $\pi_{2n}(X)$ is finite.

**Corollary to Theorem 1.** Kernel $R$ is finite.

$X_n$ is a one-point union of (at least two) $n$-spheres, so that $H_n(X_n) = H_n(X)$ is finitely generated free abelian. Moreover, it is easy to show that the homology functor $H_n$ takes $\mathcal{E}(X_n)$ isomorphically onto the group of automorphisms of $H_n(X)$. We call this automorphism group $\text{Aut}(H_n(X))$.

Let $\mu: H_n(X) \otimes H_n(X) \rightarrow \mathbb{Z}$ be the integral bilinear form determined by the intersection pairing on $H_n(X)$. Wall [8] shows that $\mu$, together with a certain function $H_n(X) \rightarrow \pi_{2n-1}(S^n)$, completely deter-
mines the homotopy type of $X$. For algebraic convenience, we modify this function slightly, obtaining a homomorphism $c$ on $H_n(X)$, which together with $\mu$ also determines the homotopy type of $X$. We do not define $c$ here.

Let $\text{Aut}(\mu, c)$ be the subgroup of $\text{Aut}(H_n(X))$ consisting of all automorphisms that preserve $c$ and that, up to sign, preserve $\mu$.

**Theorem 2.** The functor $H_n$ maps image $R$ isomorphically onto $\text{Aut}(\mu, c)$.

**Theorem 3.** $\text{Aut}(\mu, c)$ is finitely generated. If $n$ is even and $\mu$ is a definite quadratic form, or if $n$ is even and $\mu$ has rank two and index zero, then $\text{Aut}(\mu, c)$ is finite. Otherwise, $\text{Aut}(\mu, c)$ is infinite.

Combining Theorems 2 and 3 with the fact that kernel $R$ is finite, we obtain the following:

**Corollary to Theorem 3.** Theorem 3 holds for $\mathcal{S}(X)$ in place of $\text{Aut}(\mu, c)$.

Let $\mathcal{D}(X)$ be the subgroup of $\mathcal{S}(X)$ consisting of all classes represented by diffeomorphisms $X \to X$.

**Theorem 4.** Suppose that $n^2 \equiv 2 \pmod{4}$, $n \neq 2$. There is a number $k$, depending only on $n$ and on rank $(H_n(X))$, such that the index of $\mathcal{D}(X)$ in $\mathcal{S}(X)$ is less than $k$.

**Corollary to Theorem 4.** Under the above restriction on $n$, Theorem 3 holds for $\mathcal{D}(X)$ in place of $\text{Aut}(\mu, c)$.

**Examples.**

(a) Let $CP^n$ be complex projective $n$-space. Using the exact sequence of Theorem 1, together with well-known facts about the homotopy type of $CP^2$, it is easy to calculate that $\mathcal{S}(CP^n) \cong \mathbb{Z}_2$.

Indeed, an easy but unrelated argument shows that $\mathcal{S}(CP^n) \cong \mathbb{Z}_2$, for all $n \geq 1$.

(b) Let $KP^n$ be quaternion projective $n$-space. Using Theorem 1 again, together with certain accessible but less well-known facts about the homotopy type of $KP^2$, one may calculate that $\mathcal{S}(KP^n) \cong \mathbb{Z}_2$.

In contrast to the above example, however, image $R$ here is trivial. This implies:

**Proposition 1.** Every homotopy equivalence $f: KP^n \to KP^n$, $n \geq 2$, induces the identity automorphism of cohomology.

(c) We determine $\mathcal{S}(S^n_1 \times S^n_2)$, $n \geq 2$. In this case, $X_n$ is the one-point union $S^n_1 \vee S^n_2$. We need some notation:
$i$ is the homotopy class of the inclusion $S^n_i \vee S^n_2 \to S^n_1 \times S^n_2$;
$e_i$ is the element of $\pi_n(S^n_i \vee S^n_2)$ represented by the inclusion of $S^n$ onto $S^n_i$, $i=1, 2$;
$x$ is the homotopy class of the Hopf map $S^3 \to S^2$, $S^{n-2}x$ its $(n-2)$-fold suspension;
$\iota_n$ is the homotopy class of the identity map $S^n \to S^n$;
$[\alpha, \beta]$ is the Whitehead product of homotopy classes $\alpha$ and $\beta$;
$\Delta_8$ is the dihedral group of order eight, a group on two generators $a$ and $b$ satisfying $a^4=b^2=ab^{-1}ab=1$;
Sym is the group of integral $2 \times 2$ matrices generated by
$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \text{ (cf., [3])}
$$
$\Delta$ will be the image of the homomorphism $(S^b)\,^{*+\mathcal{P}}$ of Theorem 1.

**Proposition 2.** (i) $\Delta$ is trivial if $n=2, 6$ or $n \equiv 3 \pmod{4}$. Otherwise $\Delta \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and is generated by
$$
\iota \circ e_1 \circ [S^{n-2} x, \iota_n] \text{ and } \iota \circ e_2 \circ [S^{n-2} x, \iota_n].
$$
(ii) If $n$ is odd, image $R \cong \text{Sym}$, whereas if $n$ is even, image $R \cong \Delta_8$.
(iii) The following sequence is split-exact:
$$
0 \to \pi_{2n}(S^n_1 \times S^n_2)/\Delta \xrightarrow{\partial} \varepsilon(S^n_1 \times S^n_2) \xrightarrow{R} \text{image } R \to 0.
$$

The action of Sym or $\Delta_8$ on $\pi_{2n}(S^n_1 \times S^n_2)/\Delta$ can be computed explicitly, so that in the range of values of $n$ for which $\pi_{2n}(S^n)$ is known, $n \geq 2$, $\varepsilon(S^n_1 \times S^n_2)$ can be completely determined.

(d) We present an example of a $(4k-1)$-connected $8k$-manifold $M, k \geq 2$, such that the index of $\mathcal{D}(M)$ in $\varepsilon(M)$ is $\geq 8$.

Choose any of the manifolds $M$ constructed in [4] such that (i) $M$ is homotopically equivalent to $S^{4k}_1 \times S^{4k}_2$; (ii) the Pontrjagin class $p_k(M)=ae_k^*+be_k^*$, where $0 \neq a \neq \pm b \neq 0$ and $e_k^*$ is the generator of $H^{4k}(M)$ corresponding, via the given homotopy equivalence, Poincaré duality, and the Hurewicz isomorphism, to the homotopy class $e_k$ described in (c), $l=1, 2$.

It is easy to show that, of all the members of image $R \cong \Delta_8$, only the identity induces an automorphism of cohomology that keeps $p_k(M)$ fixed. Since diffeomorphisms induce cohomology isomorphisms that keep Pontrjagin classes fixed, $R(\mathcal{D}(M))$ is trivial, from which the result follows.
2. Description of the homomorphisms and of the proof of Theorem 1.

DEFINITION OF $R$: $\mathcal{E}(X) \to \mathcal{E}(X_n)$. $R(f)$ is the homotopy class of the restriction to $X_n$ of any cellular representative of $f$. J. H. C. Whitehead's Cellular Approximation Theorem implies that $R$ is well-defined.

DEFINITION OF $\rho$: $\pi_{2n}(X) \to \mathcal{E}(X)$. As a CW-complex, $X = X_n \cup e^{2n}$, where the cell $e^{2n}$ is attached to $X_n$ by a map $b: S^{2n-1} \to X_n$. Therefore, we may identify $X$ with the reduced mapping cone of $b$. Pinching together all points halfway up the cone, we obtain $S^{2n} \vee X$ and a projection $\pi: X \to S^{2n} \vee X$. Given any $a: S^n \to X$, it determines a map $(a \vee 1) \circ \pi: X \to X$, where $1$ is the identity map of $X$. Passing to homotopy classes, the association $a \to (a \vee 1) \circ \pi$ determines the homomorphism $\rho$ (cf. [1], and [2, p. 179]).

DEFINITION OF $(S\alpha)^*$: $[SX_n, X] \to \pi_{2n}(X)$. $b: S^{2n-1} \to X_n$ is the attaching map of $e^{2n}$, as above, $Sb$ is its suspension, and $(S\alpha)^*$ is determined by right composition with $Sb$.

DEFINITION OF $\psi$: $[SX_n, X] \to \pi_{2n}(X)$. We introduce notation analogous to that of example (c), above:

- $i$ is the homotopy class of the inclusion $X_n \subset X$;
- $e_k$ is the homotopy class of the inclusion of $S^n$ onto the $k$th sphere of the one-point union of $n$-spheres $X_n$;
- $S\alpha$ is the suspension of $\alpha$, and $[\alpha, \beta]$ is the Whitehead product of $\alpha, \beta$;
- $(\Gamma_{i,k})$ is the unimodular matrix determined by the cup product of $H^*(X)$ with respect to the basis of $H^*(X) = \text{Hom}(H_n(X), \mathbb{Z})$ dual to $\{e_1, e_2, \cdots \} \subset \pi_n(X_n) = H_n(X_n) = H_n(X)$. Then, we define $\psi$ by

$$
\psi(x) = \sum_{i,k} \Gamma_{i,k} [x \circ S e_i, i \circ e_k].
$$

$\psi$ arises roughly because of the failure of right composition with $b$ to determine a homomorphism $\pi(X_n, X) \to \pi_{2n-1}(X)$.

REMARKS ON THE PROOF OF THEOREM 1. The proof of Theorem 1 is an easy obstruction-theoretic exercise until one gets to proving exactness at $\pi_{2n}(X)$. At this point it is necessary to characterize a certain obstruction set (see [2, p. 185]). It is not at all difficult to show that this set is some homomorphic image of $[SX_n, X]$. The difficulty lies in showing that the homomorphism is $(S\alpha)^* + \psi$.

The arguments in this proof can be generalized. However, in general, image $R$ will not have so simple a description as that supplied by Theorem 2.
THE SOLUTION BY ITERATION OF LINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

BY F. E. BROWDER AND W. V. PETRYSHYN

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Let $X$ be a Banach space (real or complex), $T$ a bounded linear operator from $X$ to $X$. We are concerned with the solution of the equation

\[ (1) \quad u - Tu = f, \]

by the iteration process of Picard-Poincaré-Neumann,

\[ (2) \quad x_{n+1} = Tx_n + f \quad (x_0 \text{ given}), \]

i.e. with the convergence of the sequence

\[ x_n = T^n x_0 + (f + Tf + \cdots + T^{n-1} f). \]

By an earlier result of the first-named author (Browder [2]), if $X$ is reflexive, a solution $u$ for the equation (1) will exist for a given element $f$ of $X$ and an operator $T$ which is asymptotically bounded (i.e. $\|T^k\| \leq M$ for some $M > 0$ and all $k \geq 1$) if and only if the sequence $\{x_n\}$ is bounded for any fixed $x_0$. Our object in the present paper is to

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