AN INEQUALITY CONCERNING MEASURES

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Communicated by G. A. Hedlund, February 17, 1966

If \( \mu \) is a complex measure (countably additive on a \( \sigma \)-field of sub­­sets of some space), it is obvious that there is a measurable set \( E \) such that

\[
| \mu(E) | \geq \frac{1}{4} \| \mu \|
\]

where \( \| \mu \| \) denotes the total variation of \( \mu \). In fact a set \( E \) can be found for which

\[
| \mu(E) | \geq \frac{1}{\pi} \| \mu \|.
\]

We shall give a simple proof of this. If \( \mu \) is a vector valued measure with values in \( \mathbb{R}^n \) (with the usual Euclidean norm) we shall show by a suitable modification of our argument that there is a set \( E \) with

\[
\| \mu(E) \| \geq \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n + 1)/2)} \| \mu \|.
\]

Asymptotically this is \( \| \mu \|/(2\pi n)^{1/2} \), which is much better than the obvious \( \| \mu \|/2\pi n \).

**Theorem 1.** Let \( \mu \) be a complex valued measure of total variation 1. Then there is a measurable set \( E \) such that

\[
| \mu(E) | \geq 1/\pi.
\]

**Proof.** Consider first the special case where \( \mu \) is a Borel measure on the unit circle of the complex plane (which we identify with the real line (mod 2\( \pi \))), and is such that for every measurable set \( E \),

\[
\mu(E) = \int_E e^{i\theta} \mu(\theta) \, d\theta
\]

where \( \mu(E) \) denotes the total variation of \( \mu \) on the set \( E \). Then

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1 Partially supported under grant NSF-GP-5493.
AN INEQUALITY CONCERNING MEASURES

\[
\max_{E \text{ measurable}} |\mu(E)| = \max_{E \text{ measurable}} \left| \int e^{i\theta} |\mu| (d\theta) \right|
\]

\[
\geq \max_{\lambda} \left| \int_{\lambda - \pi/2}^{\lambda + \pi/2} e^{i\theta} |\mu| (d\theta) \right| = \max_{\lambda} \left| \int_{\lambda - \pi/2}^{\lambda + \pi/2} e^{i(\theta - \lambda)} |\mu| (d\theta) \right|
\]

\[
\geq \max_{\lambda} \int_{\lambda - \pi/2}^{\lambda + \pi/2} \Re(e^{i(\theta - \lambda)}) |\mu| (d\theta)
\]

\[
\geq \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\lambda - \pi/2}^{\lambda + \pi/2} \Re(e^{i(\theta - \lambda)}) |\mu| (d\theta) d\lambda
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\lambda - \pi/2}^{\lambda + \pi/2} \Re(e^{i(\theta - \lambda)}) d\lambda |\mu| (d\theta) = \frac{1}{\pi}.
\]

For the general case define \( f \) to be the Radon-Nikodym derivative \( f = d\mu/d|\mu| \), and define \( \nu(E) = \mu(f^{-1}(E)) \) for \( E \) a Borel subset of the unit circle. The proof is easily completed by application of the special case to the measure \( \nu \).

The constant \( 1/\pi \) is best possible; for some measures \( \mu \) there is no set \( E \) with \( |\mu(E)| > 1/\pi \). We shall now determine these measures.

**Theorem 2.** Let \( \mu \) be a complex valued measure with \( ||\mu|| = 1 \), and \( f \) the Radon-Nikodym derivative \( d\mu/d|\mu| \). Then a necessary and sufficient condition that there be no measurable set \( E \) with \( |\mu(E)| > 1/\pi \) is that

\[
\int f(t)^n |\mu| (dt) = 0
\]

for \( n = \pm 1, \pm 2, \pm 4, \pm 6, \pm 8, \pm \cdots \).

**Proof.** Define \( F_{\lambda} = \{ t; \lambda - \pi/2 \leq \arg f(t) \leq \lambda + (\pi/2) (\text{mod } 2\pi) \} \).

If \( E \) is any measurable set, \( \mu(E) = re^{i\lambda} \) for some choice of real numbers \( r > 0 \) and \( \lambda \); it is then easily checked that \( \Re(e^{-i\lambda} \mu(F_\lambda)) \geq r \). Thus \( |\mu(E)| \leq 1/\pi \) for all measurable sets \( E \) if and only if \( \Re(e^{-i\lambda} \mu(F_\lambda)) \leq 1/\pi \) for all real \( \lambda \). As in the proof of Theorem 1, we observe that \( f \) induces a measure \( \nu \) on the unit circle such that \( \nu(S) = \mu(f^{-1}(S)) \) for each measurable set \( S \) of the unit circle. Then

\[
\Re(e^{-i\lambda} \mu(F_\lambda)) = \int_{\lambda - \pi/2}^{\lambda + \pi/2} \Re(e^{i(\theta - \lambda)}) |\nu| (d\theta).
\]

But this is a continuous function of \( \lambda \) whose mean for \( 0 \leq \lambda \leq 2\pi \) was shown in the proof of Theorem 1 to be \( 1/\pi \). Thus it never exceeds \( 1/\pi \) in value if and only if it is constant and a continuous function on the interval \( [0, 2\pi] \) is constant if and only if its nonzero
Fourier coefficients vanish. Moreover we may interpret the function $\text{Re}(e^{-i\lambda F_0})$ as the convolution of the measure $|\nu|$ with the function defined to be $\text{Re}(e^{i\lambda})$ for $-\pi/2 \leq \lambda \leq \pi/2$, and zero elsewhere on the interval $[-\pi, \pi]$, and then extended to a periodic function. With this interpretation we see that $\text{Re}(e^{-i\lambda F_0})$ has vanishing nonzero Fourier coefficients if and only if the $n$th Fourier-Stieltjes coefficient of the measure $|\nu|$ vanishes for $n = \pm 1, \pm 2, \pm 4, \pm 6, \ldots$. But the $n$th Fourier-Stieltjes coefficient of $|\nu|$ is

$$\int_{0}^{2\pi} e^{in\theta} |\nu| (d\theta) = \int f(t)^n |\mu| (dt).$$

The proof is thus complete. A final remark: the vanishing of the $n$th Fourier-Stieltjes coefficients of $|\nu|$ for $n$ even, $n \neq 0$, means

$$\frac{1}{2}(|\nu| (d\theta) + |\nu| (d(\pi + \theta))) = d\theta/2\pi,$$

and thus implies that $|\nu|$ is absolutely continuous with respect to Lebesgue measure.

Professor S. Kakutani has suggested the following geometric proof of Theorem 1. The condition that $||\mu|| = 1$ is equivalent to the condition that the convex hull of the range of $F_0$ have perimeter 2, a fact which is easily seen for a finite measure space and easily deduced from this for a general measure space. (If $\mu$ is completely nonatomic its range is already a convex set, by a theorem of Liapunoff, see [2]). We thus consider the following isoperimetric problem; “Of all convex sets of perimeter 2, which one is contained in the smallest disk with centre 0?” It is easily seen that the answer is the disk of radius $1/\pi$, and from this fact Theorem 1 follows.

If $F_0$ is merely a finitely additive set function (complex valued of total variation 1) it is easily deduced from Theorem 1 (for finite measure spaces) that for any $\epsilon > 0$ there is a measurable set $E$ with

$$\mu(E) \geq 1/\pi - \epsilon.$$

It may be asked how the constant $1/\pi$ must be changed if instead of the usual Euclidean distance, the plane is given a different norm $||\cdot||$. Using the approach of Professor Kakutani it is not difficult to show that the constant becomes $2/s$, where $s$ is the perimeter of the unit ball $\{x; ||x|| \leq 1\}$, $s$ being measured with the distance function obtained from the norm $||\cdot||$. This perimeter is smallest when the unit ball is a regular hexagon: in this case the perimeter is 6.
We now consider the vector valued case.

**Theorem 3.** Let \( \nu \) be a measure with values in \( \mathbb{R}^n \) and such that \( \|\nu\| = 1 \). Then there is a measurable set \( E \) with

\[
\|\nu(E)\| \geq \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \right) / \left( 2\pi^{n/2} \right).
\]

**Proof.** We introduce the following notation. Denote by \( S \) the unit sphere in \( \mathbb{R}^n \), and by \( S^+ \) the set \( \{ x; x \in S, x_1 \geq 0 \} \) (\( x_1 \) being the first co-ordinate of \( x \)). Denote by \( m \) the usual spherical mean on \( S \); that is the uniformly distributed measure on \( S \) with \( m(S) = 1 \). Let \( G \) denote the orthogonal group acting in \( \mathbb{R}^n \). Let \( x_0 \) be the point \((1, 0, 0, \ldots, 0)\) of \( S \) and let \( K \) be the group of those elements of \( G \) which fix \( x_0 \). We shall use the notation \( m_k \) for Haar measure on \( K \), and \( m_G \) for the Haar measure on \( G \) (with the usual normalization for compact groups). For each positive measure \( \mu \) on \( S \) define a positive measure \( \bar{\mu} \) on \( G \) as follows: if \( f \) is a continuous function on \( G \) define \( f \) on \( S \) by

\[
f(gx_0) = \int_K f(gk)m_k(\,dk) \]

and define \( \bar{\mu} \) to be that measure on \( G \) such that for any continuous function \( f \) on \( G \)

\[
\int_G f(g)\bar{\mu}(\,dg) = \int_S f(x)\mu(\,dx).
\]

It is obvious that \( \bar{m} = m_G \), and that for any continuous function \( h \) on \( S \),

\[
\int_S h(x)\mu(\,dx) = \int_G h(gx_0)\bar{\mu}(\,dg).
\]

Finally, denote by \( \phi \) the continuous function on \( S \) defined by \( \phi(x) = \max(x_1, 0) \). As in the proof of Theorem 1 there is no loss of generality in assuming that the measure \( \nu \) is a Borel measure on \( S \) such that

\[
\nu(E) = \int_E x\mu(\,dx)
\]

for each measurable set \( E \), where \( \mu \) is a probability measure on \( S \). But then
max \[ \int_E x\mu(dx) \leq \max_{\varphi \in \mathcal{G}} \left( \int_{\varphi^{-1}S^+} \varphi(x)\mu(dx) \right) \]

\[ = \max_{\varphi \in \mathcal{G}} \left( \int_{\varphi^{-1}S^+} \varphi(x)\mu(dx) \right) \]

\[ \geq \max_{\varphi \in \mathcal{G}} \int_{\varphi^{-1}S^+} (gx)^\mu(dx) = \max_{\varphi \in \mathcal{G}} \int_{\varphi^{-1}S^+} \phi(gx)\mu(dx) \]

\[ = \max_{\varphi \in \mathcal{G}} \int_S \phi(gx)\mu(dx) = \max_{\varphi \in \mathcal{G}} \int_S \phi(gg'x_0)\mu(dg') \]

\[ \geq \int_S \int_S \phi(gg'x_0)\mu(dg')m_\mathcal{G}(dg) = \int_S \int_S \phi(gg'x_0)m_\mathcal{G}(dg)\mu(dg') \]

\[ = \int_S \int_S \phi(gx_0)m_\mathcal{G}(dg)\mu(dg') = \int_S \phi(gx_0)m_\mathcal{G}(dg) \]

\[ = \int_S \phi(x)m(dx) = \int_{S^+} x_1m(dx) = \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}. \]

As in Theorem 1 this is best possible, as the case \( \mu = m \) demonstrates. After the obvious modification the discussion after Theorem 2 on the case of finitely additive set functions is applicable once again.

There seems to be no satisfactory geometric proof of Theorem 3 analogous to the one suggested by Professor Kakutani for Theorem 1. However the condition \( \|v\| = 1 \) can be stated geometrically in terms of the convex hull of the range of \( v \). It is known that if \( K \) is any compact convex set, and if \( B \) denotes the unit ball of \( R^n \), then the volume of \( K + rB \) is a polynomial in \( r \) of degree \( n \) (see [1]). If \( K \) is the convex hull of the range of the vector valued measure \( \nu \), then the condition that \( \|v\| = 1 \) is equivalent to the coefficient of \( r^{n-1} \) in the polynomial \( \text{vol}(K + rB) \) being equal to the \( n-1 \) dimensional volume of the unit ball in \( R^n \).

**References**


University of Illinois and
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