ON CERTAIN BISIMPLE INVERSE SEMIGROUPS

BY R. J. WARNE

Communicated by O. G. Harrold, February 21, 1966

If $S$ is a semigroup, $E_S$ will denote the collection of idempotents of $S$. A bisimple semigroup $S$ is called I-bisimple if and only if $E_S = \{ e_i : i \in I, \text{the integers} \}$ with $e_i \leq e_j$ if and only if $i \geq j$. We announce the determination of the structure of I-bisimple semigroups mod groups and a determination of several of their properties. We also give a certain generalization of the bicyclic semigroup and indicate an application of this result. We use the notation and terminology of [2].

THEOREM 1. $S$ is an I-bisimple semigroup if and only if $S \cong G \times I \times I$ under the multiplication

$$(g, n, m)(h, p, q) = (g \alpha^{p-r} h \alpha^{m-r}, n + p - r, m + q - r)$$

where $r = \min(m, p)$, $\alpha$ is an endomorphism of $G$, and $\alpha^0$ is the identity transformation or equivalently

$$(g, n, m)(h, p, q) = (g \alpha^{s-m-p} h \alpha^{s-n}, n + p, s)$$

where $s = \max(m + p, q)$.

PROOF. [9, Theorem], [1, Main Theorem], [8, Theorem 1.2 and Theorem 2.2] and [5, Theorem 3.3] are important.

REMARK. An I-bisimple semigroup $S$ has no identity and hence its structure may not be obtained by specializing the Clifford structure theorem [1]. $S$ is a union of a chain of bisimple (inverse) semigroups $S_i (i \in I)$ with identity such that $E_{S_i} = \{ e_i : i \in I^0, \text{the non-negative integers} \}$ with $e_i \leq e_j$ if and only if $i \geq j$. The structure of these semigroups was given mod groups by Reilly [6] and Warne [11]. Warne obtained the result by specializing the Clifford structure theorem [1]. Incidently, the multiplication is given by (1) with $I^0$ replaced for $I$.

If $S$ is an I-bisimple semigroup with structure group $G$ and structure endomorphism $\alpha$, we will write $S = (G, \alpha)$.

Let $N$ denote the natural numbers.

THEOREM 2. Let $S = (G, \alpha)$ and $S^* = (G^*, \beta)$. Let $\{ f_i : i \in I \setminus N \}$ be a

1 These structure theorems represent a next stage in the development of bisimple semigroups to the Rees Theorem in that the determination is complete (mod groups).

2 The structure of bisimple (inverse) semigroups such that $E_S$ is linearly ordered has been given mod bisimple inverse semigroups with identity by Warne [9].

679
collection of homomorphisms of \( G \) into \( G^* \), \( \{ X_i : i \in \mathbb{N} \} \) be a collection of nondecreasing functions of \( I \) into \( I \), \( a \in I^0 \), and \( \{ z_i : i \in \mathbb{N} \} \) be a collection of elements of \( G^* \) such that

1. \( x_{C_{z_i}} = z_i x_{\bar{z}_i}^{-1} \) for \( x \in G^* \),
2. \( f \beta^a C_{z_i} = \alpha f_i \),
3. \( z_i \beta^a = z_{i+1} \), and
4. \( X_{i+1} = X_i + a \).

For each element \( (g, x, y) \in G, Se_i(i \in \mathbb{N}) \) define \( (g, x, y) \theta = [z_i \beta^a(\alpha - i)] \ldots \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \) \( \cdot [z_i \beta^a(\alpha - i)] \). If \( x(y) = i \), the left (right) multiplier of \( g_f \) \( \in \) \( \mathfrak{g} \), the identity of \( G^* \). The square brackets indicate an element of \( S^* \). Then, \( \theta \) is a homomorphism of \( S \) into \( S^* \) and conversely every homomorphism of \( S \) into \( S^* \) is obtained in this fashion. \( S \equiv S^* \) if and only if each \( f_i \) is an isomorphism of \( S \) onto \( S^* \) and (1), (2), and (3) are valid with \( a = 1 \).

**Proof.** The proof involves an application of \[8, \text{Theorem 2.3, Theorem 1.1, and Theorem 1.2}\].

Every congruence \( \rho \) on an \( I \)-bisimple semigroup \( S = (G, \alpha) \) is either a group congruence \( (S/\rho \alpha \text{ is a group}) \) or an idempotent separating congruence \( (\rho \text{-class contains at most one idempotent}) \). The group congruences are uniquely determined by the normal subgroups of the maximal group homomorphic image of \( S \). \( \rho \) is idempotent separating if and only if \( \rho = p^\rho ((g, a, b)p^\rho (h, c, d) \text{ if and only if } a = c, b = d, \text{ and } V_g = V_h \text{ where } V \text{ is a subgroup of } G \text{ such that } h(\alpha a)h^{-1} \in V \text{ for } h \in G, g \in V, \text{ and } n \in I^0 \). Results of \[4\] are significant here.

**Theorem 3.** Let \( S = (G, \alpha) \) and let \( e \) be the identity of \( G \). If \( N = \{ g \in G / g \alpha^n = e \text{ for some } n \in I^0 \} \), \( N \) is a normal subgroup of \( G \). Let \( g \rightarrow \bar{g} \) be the natural homomorphism of \( G \) onto \( G/N \). If \( (xN) \theta = (x \alpha) N \), \( x \in G \), \( \theta \) is an endomorphism of \( G/N \). The maximal group homomorphic image \( H \) of \( S \) is isomorphic to \( G/N \times I \) under the definition of equality, \( (g, b-a) = (h, d-c) \), \( h, \bar{g} \in G/N, a, b, c, d \in I^0 \) if there exist \( x, y \in I^0 \) such that \( x+b = y+d, x+a = y+c, \text{ and } \bar{g} \theta = h \theta \) and the multiplication \( (g, b-a)(h, d-c) = (g \theta^a h \theta^b, (b+d)-(a+c)) \). The homomorphism of \( S \) onto \( H \) is given by \( (g, i+a, i+b) \theta = (g, b-a) \text{ where } i \in I, a, b \in I^0 \).

**Proof.** We utilize \[7, \text{pp. 431–434, especially Theorem 2.1}\]. q.e.d.

If \( \sigma \) is the minimum group congruence on \( S = (G, \alpha), S/\mathfrak{I} \cap \sigma = (G/N, \theta) \) \( (\theta, N \text{ are defined in the statement of Theorem 3}) \) and by \[3\] \( S/\mathfrak{I} \cap \sigma = (I, +) \).

To determine the (ideal) extensions of \( S = (G, \alpha) \) by an arbitrary semigroup \( T \), one utilizes the translational hull \( \mathcal{S} \) of \( S \) \[2, \text{p. 140}\].

**Theorem 4.** Let \( S = (G, \alpha) \) and \( M = (I, G) \) be the full group of mappings of \( I \) into \( G \) (pointwise multiplication). \( H = \{ \beta \in M(I, G)/(i+1) \beta = (i \beta) a \text{ for all } i \in I \} \) is a subgroup of \( M(I, G) \). Let \( \rho_i (i \in I) \) be the inner
right translation of \((I, +)\) determined by \(i\). Thus, \(W = H \times I\) under the multiplication \((\beta, i) \cdot (\gamma, j) = (\beta \circ \rho \gamma, i + j)\) where \(\circ\) is the operation in \(H\) and juxtaposition denotes iteration of mappings is a group. Then \(S = W \cup S\) with multiplication \((\beta, a) \cdot (g, i, j) = ((i - a) \beta \cdot g, i - a, j)\) and \((g, i, j) \cdot (\beta, a) = (g(j \beta), i, j + a)\) for \((S \cap W = \emptyset)\).

**Corollary.** Every extension of \(S = (G, \alpha)\) by \(T = M^0(G^*; K, \Lambda; P) \cdot (T = M^0(G^*; K, K; \Delta))\) is given by a partial homomorphism \([15, p. 522]\) if \(T\) has proper divisors of zero.

**Theorem 5.** Let \(S = (G, \delta)\) and \(T = M^0(G^*; K, \Lambda, P)\). Let the following functions be given: \(\psi: K \rightarrow I, \theta: \Lambda \rightarrow I, \alpha: K \rightarrow G, \beta: \Lambda \rightarrow G, \gamma \) a homomorphism of \(G^*\) into \(G\) such that \(\psi \delta \neq 0\) implies \(\theta \delta = i\psi\) and \((\lambda \delta)(i \alpha) = \psi \delta \gamma\). Then \(\phi\) defined on \(T^*\) by \(\star(a; i, j) = ((i \alpha)(a \gamma)(j \beta); i \psi, j \psi)\) is a partial homomorphism of \(T^*\) into \(S\) and conversely every partial homomorphism of \(T^*\) into \(S\) is obtained in this fashion. If \(T = M^0(G^*; K, K; \Delta)\) \(\ast\) becomes \((a, i, j) = ((i \alpha)(a \gamma)(j \alpha)^{-1}, i \psi, j \psi)\).

In the case \(T^* = M(R; K, \Lambda, P)\) is completely simple one may give an explicit determination of the extensions of \(S\) by \(T\) in terms of a homomorphism of \(R\) into \((I, +)\), mappings of \(R \rightarrow H\) (see statement of Theorem 4), \(K \rightarrow H, K \rightarrow I, \Lambda \rightarrow H,\) and \(\Lambda \rightarrow I\) or by partial homomorphisms \([14]\).

We next give a certain generalization of the bicyclic semigroup, \(C\). Let \(C \circ C\) denote \(C \times C\) under the multiplication \(((m, n), (k, t)) \cdot ((m', n'), (k', t')) = ((m, n)(m', n'), f(n, m'))\) where \(f(n, m') = (k, t, (k, t)(k', t'))\), or \((k', t')\) according to whether \(n > m', n = m',\) or \(n < m'\). (See \([10]\).) \(E_S\) is lexicographically ordered if and only if \(E_S\) is order isomorphic to \(I^0 \times I^0\) under the order \((n, m) < (k, s)\) if \(k < n\) or \(k = n\) and \(m > s\).

**Theorem 6.** Let \(S\) be a bisimple semigroup. \(E_S\) is lexicographically ordered if and only if \(3C\) is a congruence on \(S\) and \(S/3C \cong C \circ C\). If \(S\) has a trivial group of units \(S \cong C \circ C\).

**Proof.** \([5, \text{Theorem 2.2}]\) and \([8, \text{Theorem 1.2}]\) are relevant.

The above definitions and theorems may be generalized to arbitrary finite dimensions. For a class of bisimple semigroups \(S\) such that \(E_S\) is lexicographically ordered, \(S \cong G \times C \circ C\) where \(G\) is a certain group under a suitable multiplication \([11]\).

Warne \([8]\), \([11]\) discussed the structure of bisimple inverse semigroups with identity on which \(3C\) is a congruence. This is the case for all semigroups given here. However, let \(F\) be the positive part of any ordered field and let \(P = (F \circ) \times F\) under the multiplication
\((a, b)(c, d) = (ac, bc+d)\). If we substitute \(P\) in the Clifford construction \([1]\), we obtain a bisimple inverse semigroup with identity on which \(\mathcal{C}\) is not a congruence.

The results given here will appear in \([11–14]\).

*Added in proof.* In \([17]\), we give examples of bisimple inverse semigroups without identity on which \(\mathcal{C}\) is not a congruence and the lexicographic case (with and without identity) is developed fully in \([16]\) and \([17]\).

**References**


*West Virginia University*