NUCLEARITY IN AXIOMATIC POTENTIAL THEORY[1]

BY BERTRAM WALSH AND PETER A. LOEB

Communicated by R. Arens, March 25, 1966

1. **Introduction.** The axiomatic approach to potential theory instituted by Brelot [3] is well known; it abstracts in an elegant manner the properties of harmonic functions which underlie much of classical potential theory and—from a more utilitarian viewpoint—reduces the question of determining whether elliptic equations with certain classes of coefficients have a boundary-value problem theory resembling that of the Laplace equation to a few more tractable questions of an essentially local character. The purpose of this note is to announce that certain properties of the linear spaces of solutions of elliptic equations with highly differentiable coefficients (properties which bear on the construction of kernel functions) are also present in the axiomatic settings, as are certain types of behavior at ideal boundaries. We list only the main results; subsidiary results and proofs will be given elsewhere.

2. **Nuclearity.** Establishment of the nuclearity results does not require the full power of the Brelot axioms. Let W be a locally compact Hausdorff space, and let there be given a set $\mathcal{H}$ of continuous real-valued functions on W satisfying the following (sheaf) axiom:

I. The domains of elements of $\mathcal{H}$ are open subsets of W; each $f\in\mathcal{H}$ is continuous on its domain; for fixed open $\Omega\subseteq W$ the set $\mathcal{H}_\Omega=\{f\mid f\in\mathcal{H}$, Domain $(f)=\Omega\}$ is a real vector space, and a function g with open domain $\Omega\subset W$ belongs to $\mathcal{H}$ iff for each $x\in\Omega$, $\exists h\in\mathcal{H}$ and open ω with $x\in\omega\subseteq\Omega$ such that $g|_\omega=h|_\omega$.

Given two classes $\mathcal{H}$ and $\mathcal{K}$ satisfying I above, we shall say $\mathcal{K}$ is a subclass of $\mathcal{H}$ iff $\mathcal{K}\subseteq\mathcal{H}$; this is equivalent to saying that for any open $\Omega\subseteq W$ the vector space $\mathcal{K}_\Omega$ is a subspace of $\mathcal{H}_\Omega$.

For open $\Omega\subseteq W$ let $\mathcal{H}_\Omega{}^c$ denote the vector subspace of $\mathcal{H}_\Omega$ consisting of those functions which have (necessarily unique) continuous extensions to $\bar{\Omega}$. A relatively compact open subset Ω will be said to be *regular* (with respect to $\mathcal{H}$) if there exists a subset $\delta\Omega\subseteq\bar{\Omega}$ with the property that the restriction mapping $f\to f|_{\delta\Omega}$ of $\mathcal{H}_\Omega^c$ into $\mathcal{C}(\delta\Omega)$ is an order-preserving isomorphism (onto); while the uniqueness of $\delta\Omega$ is not important in what follows, conditions can be given which insure

[1] The preparation of this paper was sponsored in part by N.S.F. Grants GP-4563 and GP-5279 respectively.

that it is a uniquely determined subset of the Šilov boundary of $\bar{\Omega}$ with respect to $\mathcal{H}^c_{\bar{\Omega}}$; see [1]. The classes $\mathcal{H}$ we shall consider satisfy local regularity and monotone-completeness conditions, summarized in the following axioms:

II. There is a base for the topology of W consisting of $\mathcal{H}$-regular open sets.

III. For every open $\Omega \subseteq W$ and $x \in \Omega$ there exists a neighborhood ω with $x \in \omega \subset \Omega$, having the property that if $\mathcal{G}$ is a subset of $\mathcal{H}$ and $\mathcal{G}$ is directed upward under pointwise order, then $\sup_{\mathcal{G}} f(x) < \infty$ if and only if $(\sup_{\mathcal{G}} f)|_{\omega} \in \mathcal{H}_{\omega}$.[2]

Let it be agreed that any $\mathcal{C}(X)$, where X is a locally compact Hausdorff space, will be topologized by uniform convergence on compacta in X; then one may talk about its closed subsets. We observe that axiom II above implies that for any open $\Omega \subseteq W$, $\mathcal{H}_{\Omega}$ is closed; a class $\mathcal{H}$ for which every $\mathcal{H}_{\Omega}$ is closed will be called *complete.*

THEOREM 1. *If Ω is a regular open set for a complete class $\mathcal{H}$ satisfying* I *and* III, *then for any open Ω_1 and Ω_2 with $\Omega_2 \supseteq \bar{\Omega} \supseteq \Omega \supseteq \bar{\Omega}_1$, the restriction mapping from $\mathcal{H}_{\Omega_2}$ to $\mathcal{H}_{\Omega_1}$ is an integral linear transformation* [5, *Chapter* I, *pp.* 126–7]; *if Ω is σ-compact, then the restriction mapping $\mathcal{H}_{\Omega_2} \to \mathcal{H}_{\Omega}$ is integral.*

COROLLARY (HARNACK; cf. [8]): *If $\mathcal{H}$ satisfies* I, II *and* III, *then for any open Ω and $x_0 \in \Omega$, the set $\{f \mid f \in \mathcal{H}_{\Omega}, f \geq 0, f(x_0) = 1\}$ is equicontinuous in a neighborhood ω of x_0.*

In many interesting cases the following condition holds in certain open subsets $\Omega \subseteq W$:

IV. There exists a family of regular open subsets of Ω, directed upward under inclusion, whose union is Ω.

THEOREM 2. *If $\mathcal{H}$ satisfies* I, II *and* III *and Ω is σ-compact, or if $\mathcal{H}$ is complete and satisfies* I *and* III *while Ω satisfies* IV, *then $\mathcal{H}_{\Omega}$ is nuclear* [5, *Chapter* II, *p.* 34].

COROLLARY. *Every closed subspace of such an $\mathcal{H}_{\Omega}$ is nuclear; the dual of any metrizable closed subspace (or of any subspace if Ω is σ-compact) is nuclear.*

Recall that a locally convex space is nuclear iff every continuous linear mapping from it into a Banach space can be put in the form $x \to \sum_{i=1}^{\infty} \lambda_i \langle x, x_i' \rangle y_i$, where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, $\{x_i'\}_{i=1}^{\infty}$ is an equicontinuous subset of the dual of the domain, and $\{y_i\}_{i=1}^{\infty}$ is a sequence

[2] This assumption implies that W is locally connected. See [2].

in the range with $\|y_i\| \to 0$; mappings of this form are called nuclear. Thus

COROLLARY. *If E is a metrizable closed subspace of $\mathcal{H}_\Omega$, F a Banach space and u: $F \to E$ a continuous linear transformation, then there exists a sequence $\{\lambda_i\}_{i=1}^\infty$ with $\sum_{i=1}^\infty |\lambda_i| < \infty$, a sequence $\{h_i\}_{i=1}^\infty \subseteq E$ tending to zero in its topology (uniformly on compacta) and a sequence $\{x_i'\}_{i=1}^\infty \subseteq F'$ with $\|x_i'\| \to 0$ such that for any $f \in F$*

$$u(f) = \sum_{i=1}^\infty \lambda_i \langle f, x_i' \rangle h_i.$$

In particular if $F = L^1(\mu)$ for some measure μ on a locally compact Hausdorff space S, then there is a sequence $\{g_i\}_{i=1}^\infty \subseteq \mathcal{L}^\infty(\mu)$ (functions, not classes) tending to zero uniformly on S such that if

$$G(x, s) = \sum_{i=1}^\infty \lambda_i g_i(s) h_i(x) \qquad (x, s) \in \Omega \times S$$

then $uf(x) = \int G(x, s) f(s) d\mu(s)$. Note that $G(\cdot, s) \in E$ for all $s \in S$.

Proof follows immediately from reflexivity of E and nuclearity of E_b'.

Given a harmonic class $\mathcal{H}$ and a complete locally convex vector space F, it may be desirable to define a class of F-valued functions which belong to $\mathcal{H}$ in some sense. The weakest sense in which an F-valued function on an open $\Omega \subseteq W$ might be said to be "$\mathcal{H}$-harmonic" is that for each $y' \in F'$ the real-valued function $x \to \langle f(x), y' \rangle$ belong to $\mathcal{H}_\Omega$; on the other hand, one would want the class of $\mathcal{H}$-harmonic F-valued functions to include all functions which are uniform limits on compacta of functions of the form $x \to \sum_{i=1}^n h_i(x) \cdot y_i$, where $h_i \in \mathcal{H}_\Omega$ and $y_i \in F$. Both these senses of being $\mathcal{H}$-harmonic coalesce for $\mathcal{H}$ satisfying the hypotheses of Theorem 2 above, or more generally for complete subclasses of such classes:

PROPOSITION 1. *If $\mathcal{H}$ satisfies* I, II *and* III *and Ω is σ-compact, or if $\mathcal{H}$ is complete and satisfies* I *and* III *while Ω satisfies* IV, *then for any subspace $E \subset H_\Omega$ and complete locally convex F a function f: $\Omega \to F$ is a uniform limit on compacta in Ω of functions of the form $x \to \sum_{j=1}^n h_j(x) y$, if and only if for every $y' \in F'$ the function $x \to \langle f(x), y' \rangle$ belongs to E.*

In the axiomatic potential theory of Brelot, W is taken to be non-compact, connected and locally connected; every point is assumed to have a neighborhood basis of connected open sets which satisfy II with $\delta\Omega = \partial\Omega$, the topological boundary of Ω. I and III are always assumed; IV is implied by the requirement that there exist a positive superharmonic function on Ω (see [7]). Thus Theorem 2 holds in the Brelot situation, and the corollary of Theorem 1 becomes a recent result of the authors.

3. **Ideal boundaries.** Henceforth we assume the Brelot axioms (see the paragraph immediately preceding). A *compactification* of W is a compact Hausdorff space W^* containing W as a dense open subspace; a compactification W^* is said to be *resolutive* iff for every $f \in \mathcal{C}(\Delta)$, where $\Delta = W^* \sim W$, the lower envelope $H^-(f)$ of all $\mathcal{H}$-superharmonic functions v with lim inf $v \geqq f$ at Δ equals the upper envelope $H_-(f)$ of all $\mathcal{H}$-subharmonic functions u with lim sup $u \leqq f$ at Δ; $H(f)$ is the common value of $H_-(f)$ and $H^-(f)$ when they are equal. For resolutive compactifications, the second corollary of Theorem 2 becomes the following version of a recent result of M. Nakai for Riemann surfaces [9]:

PROPOSITION 2. *Let W be a locally compact, noncompact, connected and locally connected Hausdorff space, countable at ∞, and $\mathcal{H}$ a harmonic class satisfying* I, II *and* III *above with $\delta = \partial$. Let W^* be a resolutive compactification of W and ρ be harmonic measure on Δ with base point x_0,* i.e. $\rho(f) = H(f)(x_0)$. *Then there exists a function $G(x, s) = \sum_{i=1}^{\infty} \lambda_i h_i(x) g_i(s)$ with $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, $\{g_i\}_{i=1}^{\infty}$* in $\mathcal{L}^\infty(\rho)$ *tending to zero uniformly on Δ, $\{h_i\}_{i=1}^{\infty}$ in $\mathcal{H}_W$ tending to zero uniformly on compacta in W, such that $H(f)(x) = \int f(s) G(x, s) d\rho(s)$ for each $f \in \mathcal{C}(\Delta)$. Moreover, there exists an increasing sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of Δ for which $\bigcup_{n=1}^{\infty} K_n$ has ρ-null complement and such that G is continuous on $W \times K_n$ for each n. Any two such G differ at most on $W \times N$, where N is ρ-null.*

Now we suppose, in addition to the Brelot axioms, that the function 1 is $\mathcal{H}$-superharmonic, and consider the Q-compactification [4, p. 96 ff.][3] W^* of W with $Q = \mathcal{BH}$, the (Banach) space of bounded functions in $\mathcal{H}_W$; W^* is determined up to homeomorphism by the properties: W^* is a compactification of W, each $h \in \mathcal{BH}$ has a unique extension in $\mathcal{C}(W^*)$, and the Banach subspace of $\mathcal{C}(W^*)$ consisting of those extensions separates points of $\Delta = W^* \sim W$. One may now show that the cone of positive elements of $\mathcal{BH}$ has a Šilov boundary $\kappa \subseteq W^*$, that $\kappa \subseteq \Delta$, and that $\mathcal{BH}|_\kappa = \mathcal{C}(\kappa)$. Using a generalization of a barrier argument developed in [6, pp. 44–45] we show that the

[3] [4] treats Q-compactifications only for Riemann surfaces; however, it is easy to verify that their construction and characterization make use only of the local compactness of the surface.