A PROPERTY OF THE $L_2$-NORM OF A CONVOLUTION

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Introduction. It is known that the convolution of two members, $f$ and $g$, of $L_2(-\infty, +\infty)$ can be a null function without either $f$ or $g$ being a null function. But, if one defines $f_\nu$ by setting $f_\nu(x) = e^{ix\nu}f(x)$ for all $x$, $f_\nu$, and $g$ will have a convolution that is not a null function for a suitable choice of $\nu$. There is apparently no information available on how the $L_2$-norm of the latter convolution depends on $\nu$.

A partial answer to this problem will be provided in the present paper. There will be derived a lower bound on the supremum in $\nu$ of the $L_2$-norm of the convolution of $f_\nu$ and $g$. The lower bound will be expressed in terms of a notion of $\epsilon$-approximate support which is an $L_2(-\infty, +\infty)$ analog of the concept of support of a continuous function on a locally compact space. The inequality will be shown to be sharp in the sense that one can construct an $f$ and a $g$ for which the lower bound is approached arbitrarily closely.

Definitions and notation. Because of the need for uniqueness and because of the nature of the $L_1$-norm, an appropriate analog for $L_1(-\infty, +\infty)$ of the notion of support is the following.

Definition. The $\epsilon$-approximate support of a member $f$ of $L_1(-\infty, +\infty)$ is defined to be the closed interval $I_{\epsilon,f}$ such that

(a) $I_{\epsilon,f}$ is symmetric about the smallest real number $x_0$ for which

$$\int_{-\infty}^{x_0} |f(x)| \, dx = (\frac{1}{2})\|f\|_1,$$

(b) $\int I_{\epsilon,f} |f(x)| \, dx = (1-\epsilon)\|f\|_1$, $\|f\|_1$ being the $L_1$-norm of $f$. The existence and uniqueness of $x_0$ and $I_{\epsilon,f}$ are clear from the absolute continuity of the indefinite integral of $|f|$.

For any Lebesgue-measurable set $E$ the measure of $E$ is denoted by $m(E)$ and the characteristic function is denoted by $\chi(E)$. Given any two measurable functions on the real numbers, $f$ and $g$, such that for almost all $x$, $f(y)g(x-y)$ is in $L_1(-\infty, +\infty)$ one denotes by $f * g$ the function for which $(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y) \, dy$ a.e. Given any $f$ in $L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$ one defines the Fourier transform of $f$, denoted by $\hat{f}$, by requiring that for all real $\omega$, $\hat{f}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \exp(-i\omega x)f(x) \, dx$. Thus, the definition of $\hat{f}$ for an arbitrary $f$ in $L_2(-\infty, +\infty)$ is determined.
Results. One lemma is required for proof of the principal result. It appears below.

**Lemma.** Given any two nonnegative, nonnull functions $h$ and $k$ in $L_1(-\infty, +\infty)$ such that $\|h\|_1 = \|k\|_1 = 1$, then

$$\sup_{-\infty < x < +\infty} (h * k)(x) \geq \sup_{0 < \epsilon < 1} (1 - \epsilon)^2 [m(I_{e, h}) + m(I_{e, k})]^{-1}.$$  

**Proof.** For any real $\epsilon$ such that $0 < \epsilon < 1$, we define $h_\epsilon$ and $k_\epsilon$, nonnegative and nonnull members of $L_1(-\infty, +\infty)$, by the equations below.

(2)  
$$h_\epsilon(z) = \chi_{I_{e, h}}(z)h(z), \quad \text{all } z,$$

(3)  
$$k_\epsilon(x) = \chi_{I_{e, k}}(x)k(x), \quad \text{all } x.$$

First, one can use (2) and (3) to write:

$$m(I_{e, h}) + m(I_{e, k}) = m\{x \mid I_{e, h} \cap (x - I_{e, k}) \neq \emptyset\}$$

$$= m\{x \mid \exists y \, y \in I_{e, h}, x - y \in I_{e, k}\}$$

$$\geq m\{x \mid (h_\epsilon * k_\epsilon)(x) \neq 0\}.$$  

Then since $k_\epsilon(x)h_\epsilon(y)$ belongs to $L_1(\mathbb{R})$, one can combine (4) with the Fubini theorem for multiple integrals and well-known properties of the transformation $T$ defined by $T(x, y) = (x - y, y)$ to write the following sequence of equalities.

$$[m(I_{e, h}) + m(I_{e, k})] \left[ \sup_{-\infty < x < +\infty} (h * k)(x) \right]$$

$$\geq m\{x \mid (h_\epsilon * k_\epsilon)(x) \neq 0\} \left[ \sup_{-\infty < x < +\infty} (h * k)(x) \right]$$

$$\geq m\{x \mid (h_\epsilon * k_\epsilon)(x) \neq 0\} \left[ \sup_{-\infty < x < +\infty} (h_\epsilon * k_\epsilon)(x) \right]$$

$$\geq \int_{-\infty}^{+\infty} (h_\epsilon * k_\epsilon)(x) \, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [h_\epsilon(y)k_\epsilon(x - y)] \, dxdy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [h_\epsilon(y)k_\epsilon(x)] \, dxdy$$

$$= \left( \int_{-\infty}^{+\infty} h_\epsilon(y) \, dy \right) \left( \int_{-\infty}^{+\infty} k_\epsilon(x) \, dx \right)$$

$$= (1 - \epsilon)^2.$$
The conclusion of this lemma follows directly from (5).

This lemma permits one to prove the following theorem.

**THEOREM.** Let \( f \) and \( g \) be any two members of \( L_2(\mathbb{R}) \) such that \( \| f \|_2 = \| g \|_2 = 1 \). Let \( f_\varepsilon(x) = e^{i\varepsilon x} f(x) \) for all \( x \). Let \( F(x) = |\hat{f}(x)|^2 \) and \( G(x) = |\hat{g}(x)|^2 \) for all \( x \). Then

\[
\sup_{-\infty < \varepsilon < +\infty} \| f_\varepsilon * g \|_2 \geq \sup_{0 < \varepsilon < 1} (1 - \varepsilon) \left[ 2\pi \left( m(I_{\varepsilon, 1}) + m(I_{\varepsilon, 0}) \right)^{-1} \right]^{1/2}.
\]

The inequality (6) is sharp in the sense that for every positive number \( \eta \) there are choices of \( f \) and \( g \) for which the right side of the inequality is finite and for which the ratio of the expression on the left-hand side of (6) to the expression on the right-hand side exceeds 1 by less than \( \eta \).

**PROOF.** The lemma and the Plancherel theorem combined yield (6).

To prove the rest of the theorem, let \( \eta \) be a fixed but arbitrary positive number. Then two members, \( \tilde{p} \) and \( \tilde{q} \), of \( L_2(\mathbb{R}) \) will be defined and shown to have the asserted properties relative to \( \eta \). These functions will be defined in terms of their Fourier transforms.

\[
\hat{p}(\omega) = \begin{cases} 
(\pi\Delta)^{-1/2}(1 + i\omega)^{-1}, & |\omega| \leq \tan \frac{\pi}{2}\Delta, \\
0, & |\omega| > \tan \frac{\pi}{2}\Delta,
\end{cases}
\]

and

\[
\hat{q}(\omega) = \begin{cases} 
\pi^{-1/2}(1 + i\Delta\omega)^{-1}, & |\omega| \leq \Delta^{-1}\tan \frac{\pi}{2}\Delta, \\
0, & |\omega| > \Delta^{-1}\tan \frac{\pi}{2}\Delta.
\end{cases}
\]

Here \( \Delta \) is assumed to be positive and less than 1. Then, with the aid of the definitions of \( \hat{p} \) and \( \hat{q} \) and the Plancherel theorem, it can be seen that one has:

\[
\sup_{-\infty < \varepsilon < +\infty} \| p_\varepsilon * q \|_2 = \sup_{-\infty < \varepsilon < +\infty} \left[ \frac{2\pi \int_{-\infty}^{+\infty} |\hat{p}(\omega - \nu)^2| |\hat{q}(\omega)|^2 d\omega}{2\pi \int_{-\infty}^{+\infty} |\hat{p}(\omega)|^2 |\hat{q}(\omega)|^2 d\omega} \right]^{1/2}.
\]

And the latter integral has the following evaluation.
\[
\int_{-\infty}^{+\infty} \left| \hat{p}(\omega) \right|^2 \left| \hat{q}(\omega) \right|^2 \, d\omega
\]
\[
= \left( 2\Delta^{-1} \tan \frac{\pi}{2} \Delta + 2 \tan \frac{\pi}{2} \right)^{-1} \left( \frac{\tan \pi \Delta/2}{\pi \Delta/2} \right) 
\cdot \left( 1 - \frac{2}{\pi} \tan^{-1} \left( \Delta \tan \frac{\pi \Delta}{2} \right) \right).
\]

However, one can see:

\[
m(I_0, P) = 2 \tan \frac{\pi}{2} \Delta,
\]
\[
m(I_0, Q) = 2\Delta^{-1} \tan \frac{\pi}{2} \Delta
\]

where \( P \) and \( Q \) are determined by setting \( P(\omega) = \left| \hat{p}(\omega) \right|^2 \) and \( Q(\omega) = \left| \hat{q}(\omega) \right|^2 \) for all \( \omega \). Thus, there results:

\[
\left( 2\Delta^{-1} \tan \frac{\pi}{2} \Delta + \tan \frac{\pi}{2} \Delta \right)^{1/2} \leq \sup_{0 < \epsilon < 1} \left( 1 - \epsilon \right) \left\{ 2\pi \left[ m(I_0, P) + m(I_0, Q) \right]^{-1} \right\}^{1/2}.
\]

Hence, combining (6), (9), (10), and (13), one can conclude that when \( \Delta \) is small enough for

\[
\left[ \frac{\tan \frac{\pi}{2} \Delta \cdot 1 - \frac{2}{\pi} \tan^{-1} \left( \Delta \tan \frac{\pi}{2} \Delta \right)}{\pi \Delta/2 \cdot 1 - \Delta} \right]^{1/2} - 1
\]
to be less than \( \eta \), then the same is true of

\[
\left( \sup_{-\infty < r < +\infty} \left\| p_r * q \right\|_2 \right) / \left\{ \sup_{0 < \epsilon < 1} \frac{(1 - \epsilon)(2\pi)^{1/2}}{\left[ m(I_0, P) + m(I_0, Q) \right]^{1/2}} \right\} - 1.
\]

It is, of course, clear from (9) and (10) that \( \sup_{-\infty < r < +\infty} \left\| p_r * q \right\|_2 \) is finite.

Thus, the second part of the theorem has been approved.

**Corollary.** Let the notation of the theorem hold. Further, let \( f \) and \( g \) be restrictions to \( (-\infty, +\infty) \) of entire functions of exponential type such that the types of \( f \) and \( g \) are \( E_1 \) and \( E_2 \), respectively. Then

\[
\sup_{-\infty < r < +\infty} \left\| f_r * g \right\|_2 \geq \left[ \frac{\pi}{2} \left( E_1 + E_2 \right)^{-1} \right]^{1/2}.
\]
PROOF. As indicated by Theorem 21 [1] the transforms of \( f \) and \( g \) vanish outside \([-E_1, E_1]\) and \([-E_2, E_2]\) respectively. Thus,

\[ m(I_{o,p}) \leq 4E_1, \]
\[ m(I_{o,q}) \leq 4E_2. \]

Since the indefinite integrals of \(|f|\) and \(|g|\) are absolutely continuous, (14) and (15) permit the following inequality.

\[ \sup_{0<\epsilon<1} (1 - \epsilon) \{ 2\pi [m(I_{\epsilon,p}) + m(I_{\epsilon,q})]^{-1} \}^{1/2} \geq \{ 2\pi [4E_1 + 4E_2]^{-1} \}^{1/2}. \]

The assertion of the corollary follows from (16) and the theorem.

**Reference**