F. Rellich [5] and, more generally, A. M. Molcanov [3] have shown that the problem
\[ \frac{1}{2} \Delta^2 u(x) + \lambda u(x) = 0, \quad x \in \Omega \]
\[ u(x) = 0, \quad x \in \partial \Omega \]
has a discrete spectrum (and consequently a complete orthonormal system of eigenfunctions in $L_2(\Omega)$) provided that $\Omega$ is a "quasi-bounded" domain in $E_n$. A domain $\Omega$ is said to be quasi-bounded if
\[ \lim \text{dist}(x, \partial \Omega) = 0. \]
(See [1] for a proof of Molcanov's result, based on a generalization of the Kondrachoff embedding theorem for the Sobolev spaces $H_0^m(\Omega)$.) The problem of determining the asymptotic behavior of the eigenvalues of (1) has remained open (cf. [2, p. 233]).

In the present note we consider the above problem from the point of view of random processes, as described in detail for the case of a bounded domain, as well as for the case of the operator $-\frac{1}{2} \Delta^2 + V(x)$ (with $V(x) \to +\infty$ as $|x| \to \infty$) on an unbounded domain, in the papers of D. Ray [4] and M. Rosenblatt [6]. We will show that if $\Omega$ satisfies the following condition
\[ m(\Omega \cap [a < |x| < a + 1]) = O(a^{-\beta}) \]
for some $\beta > \frac{1}{2}$, then simple modifications of Ray's arguments suffice to prove discreteness of the spectrum, as well as to obtain an asymptotic formula for the eigenvalues.

We take Ray's paper [4] as a starting point. Thus (assuming a cone condition for $\Omega$, as described in Theorem 1 below) we already have a Green's function $K(x, y, t)$ corresponding to the equation

---

1 Research supported in part by the United States Air Force Office of Scientific Research, grant number AF-AFOSR-379-65.
and zero boundary conditions. We first wish to verify that the integral operator $K_t$ with kernel $K(x, y, t)$ is completely continuous in $L_2(\Omega)$. As in [4, Lemma 3] we see that it is sufficient to show, for fixed $t > 0$,

$$\int_{\Omega \cap \{ |x| > a \}} |K \psi(x)|^2 \, dx \to 0 \quad \text{as } a \to \infty,$$

uniformly for $\psi \in L_2(\Omega), \| \psi \| = 1$. But, as in [4],

$$\int_{\Omega \cap \{ |x| > a \}} |K \psi(x)|^2 \, dx \leq \int_{\Omega \cap \{ |x| > a \}} dx \cdot \text{prob}\{x + x(t) \in \bar{\Omega}, 0 \leq \tau \leq t\} \| \psi \|^2.$$

By an elementary calculation using (2), we have for any $\beta' < \beta$

$$\text{prob}\{x + x(t) \in \bar{\Omega}, 0 \leq \tau \leq t\} \leq 2 \text{prob}\{x + x(i) \in \bar{\Omega}\} = O(|x|^{-\beta'}), \quad x \in \Omega;$$

here $t$ is fixed. Hence, writing $\Omega_\ast = \Omega \cap [i \leq |x| < i + 1], i = 0, 1, 2, \ldots$, we have (taking $\beta' > \frac{1}{2}$)

$$\int_{\Omega} \text{prob}\{x + x(\tau) \in \bar{\Omega}, 0 \leq \tau \leq t\} \, dx = \sum_i \int_{\Omega_i} \text{prob}\{x + x(\tau) \in \bar{\Omega}, 0 \leq \tau \leq t\} \, dx = O\left( \sum_i i^{-2\beta'} \right) < \infty,$$

and (4) follows from this. We therefore have

**Theorem 1.** Let $\Omega$ be an open set in $E_n$, satisfying condition (2) and the following cone condition: for each $x \in \partial \Omega$ there is an open cone with vertex $x$, lying outside $\bar{\Omega}$. Let $K_t$ be the integral operator in $L_2(\Omega)$ with kernel $K(x, y, t)$.

Then $K_t$ is completely continuous and hence has a countable set of eigenvalues $\{\exp(-\lambda_j t), j = 0, 1, 2, \ldots\}$ with corresponding complete orthonormal eigenfunctions $\{\phi_j(x)\}$, which are independent of $t$. Moreover the $\lambda_j$ are eigenvalues and the $\phi_j$ eigenfunctions of the problem (1).
Corollary. Let $\Omega$ be as in Theorem 1. Then

$$\sum_{\lambda_j < \lambda} \phi_j(x)^2 \sim \left(\frac{\lambda}{2\pi}\right)^{n/2} \frac{1}{\Gamma(1 + n/2)}$$

as $\lambda \to \infty$, for each $x \in \Omega$.

The proofs of the asserted properties of the $\lambda_j$ and $\phi_j$ are the same as in Ray's paper. In particular, Ray shows that

$$\sum_j \exp(-\lambda_j t)\phi_j(x) = K(x, x, t) \sim \left(\frac{1}{2\pi t}\right)^{n/2}$$

as $t \to 0$, uniformly for $x \in \Omega$; in the present case this follows from the fact that $K(x, y, t)$ is a Hilbert-Schmidt kernel, as can be proved in a manner similar to the above verification of (4)—it is useful to notice that $0 \leq K(x, y, t) \leq (2\pi t)^{-n/2} \exp(-|x-y|^2/2t)$.

Theorem 2. Let $\Omega \subset \mathbb{R}^n$ satisfy the hypotheses of Theorem 1. Let $\rho(x)$ be a nonnegative function in $L_1(\Omega)$. Define

$$N_\rho(\lambda) = \sum_{\lambda_j \leq \lambda} \int_{\Omega} \rho(x)\phi_j(x)^2 \, dx.$$ 

Then

$$N_\rho(\lambda) \sim \left(\frac{\lambda}{2\pi}\right)^{n/2} \frac{1}{\Gamma(1 + n/2)} \int_{\Omega} \rho(x) \, dx.$$ 

Proof. Applying (5) to the Laplace-Stieltjes transform of $N_\rho(\lambda)$, we have

$$\int_0^\infty e^{-\lambda t} \, dN_\rho(\lambda) = \int_{\Omega} \rho(x) \sum_j \exp(-\lambda_j t)\phi_j(x)^2 \, dx$$

$$= \int_{\Omega} \rho(x)K(x, x, t) \, dx$$

$$\sim \left(\frac{1}{2\pi t}\right)^{n/2} \int_{\Omega} \rho(x) \, dx.$$ 

Hence the Tauberian theorem of Karamata applies, and yields (6). q.e.d.

We obviously obtain the classical formula of Weyl if we put $\rho(x) = 1$ on a bounded region (or even on an unbounded region of finite volume). If in the general case we choose a bounded function $\rho(x)$, we obtain $N_\rho(\lambda) \leq c \cdot N(\lambda)$ where $N(\lambda) = N_1(\lambda)$ is the usual function; we
therefore obtain a one-sided estimate for $N(\lambda)$:

$$N(\lambda) \geq \lambda^{n/2}$$

where $f(\lambda) \geq g(\lambda)$ means the same as $g(\lambda) = O(f(\lambda))$. We remark that our results are unaffected if the operator $-\Delta^2$ is replaced by $-\Delta^2 + V(x)$ if $V(x)$ is a bounded function on $\Omega$.

The foregoing results can also be derived using analytical methods similar to those of Titchmarsh [7]; the basic properties of the Green's function $G(x, y, \lambda)$ in this case are due to D. Hewgill (Thesis, University of British Columbia).

**References**

5. F. Rellich, *Das Eigenwertproblem von $\Delta u + \lambda u = 0$ in Halbröhren*, in Essays presented to R. Courant, Interscience, New York, 1948, 329-344.