ON POINCARÉ’S BOUNDS FOR HIGHER EIGENVALUES

BY WILLIAM STENGERT

Communicated by A. Zygmund, February 23, 1966

1. Introduction. Let $A$ be a compact symmetric negative-definite operator on a real Hilbert space $H$ having the inner product $(u, v)$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues and $u_1, u_2, \cdots$ the corresponding orthonormal set of eigenvectors of the equation $Au = \lambda u$. Denote by $R(u)$ the Rayleigh quotient $(Au, u)/(u, u)$. For a given $\lambda_n$ let $m$ and $N$ be the smallest and largest indices respectively such that $\lambda_m = \lambda_n = \lambda_N$. There are two variational characterizations of $\lambda_n$ by inequalities. One goes back to Poincaré [1, p. 259] and was reformulated by Pólya and Schiffer [2], [3]. The other is the maximum-minimum principle for which A. Weinstein [4], [5] recently introduced a new approach. Using the Weinstein determinant and the corresponding quadratic form he gave for the first time a complete discussion of the corresponding inequalities including the necessary and sufficient conditions for equality. In the present paper we give a similar discussion of Poincaré’s characterization of $\lambda_n$.

2. The main result. Let $V_r$ be any $r$-dimensional subspace of $H$ and let $\phi_1, \phi_2, \cdots, \phi_r$ be a basis for $V_r$. We consider the determinant

$$\det \left\{ (A\phi_i, \phi_k) - \lambda (\phi_i, \phi_k) \right\}, \quad i, k = 1, 2, \cdots, r.$$  

Using Parseval’s formula we see that (1) can also be written as

$$\det \left\{ \sum_{j=1}^{\infty} (\lambda_j - \lambda) (\phi_i, u_j)(\phi_k, u_j) \right\}, \quad i, k = 1, 2, \cdots, r.$$  

Let us note in passing the remarkable, but until now unexplained, similarity between (2) and the Weinstein determinant

$$W(\lambda) = \det \left\{ \sum_{j=1}^{\infty} (\lambda_j - \lambda)^{-1} (\phi_i, u_j)(\phi_k, u_j) \right\}, \quad i, k = 1, 2, \cdots, r.$$  

We can now formulate our main result.

Theorem. For any choice of $V_r$ we have the inequality
\( \lambda_n \leq \max_{u \in V_r} R(u) \)

if and only if \( m \leq r \). By varying \( V_r \) we obtain the following characterization of \( \lambda_n \).

\[
\lambda_n = \min_{V_r} \max_{u \in V_r} R(u), \quad m \leq r \leq N.
\]

Assuming that \( m \leq r \), the necessary and sufficient conditions on the space \( V_r \) for the equality

\[
\lambda_n = \max_{u \in V_r} R(u)
\]

are that \( r \leq N \) and for any \( \epsilon > 0 \) the quadratic form with the symmetric matrix

\[
\{(A \rho_i, \rho_k) - (\lambda_n + \epsilon)(\rho_i, \rho_k)\}, \quad i, k = 1, 2, \ldots, r
\]

is negative definite.

**Proof.** The proofs of (4) and (5) have been given in [1] and [2], [3] for the case \( r = n \). Obviously (4) holds also for \( m \leq r \) since \( \lambda_m = \lambda_n \). To show the necessity of this condition we assume for the moment that (4) holds for all \( V_r \) where \( r < m \) and choose \( V_r \) to be the subspace spanned by \( u_1, u_2, \ldots, u_r \). In this case we have

\[
\max_{u \in V_r} R(u) = \lambda_r < \lambda_m = \lambda_n \leq \max_{u \in V_r} R(u)
\]

which is a contradiction. As in [2], [3] the equality (5) follows immediately not only for \( r = n \) but also for \( m \leq r \leq N \). In fact, it is sufficient to use the classical choice \( \rho_k = u_k, \quad k = 1, 2, \ldots, r \) in order to obtain (6). In §3 we give an example which shows that the classical choice is not a necessary condition for (6). To prove our necessary and sufficient conditions we shall assume that the basis \( \rho_1, \rho_2, \ldots, \rho_r \) has been chosen so that the matrix (7) is diagonal. First we show that our conditions are necessary. Suppose that (6) holds for \( r > N \). Then, using (4), we obtain the contradiction

\[
\lambda_r \leq \max_{u \in V_r} R(u) = \lambda_n = \lambda_N < \lambda_r.
\]

Since (6) implies

\[
R(\rho_i) = (A \rho_i, \rho_i)/(\rho_i, \rho_i) < \lambda_n + \epsilon, \quad i = 1, 2, \ldots, r
\]

all elements on the diagonal of (7) are negative, which proves that the quadratic form corresponding to (7) must be negative definite. To
prove sufficiency we assume that for any \( \epsilon > 0 \) the diagonal matrix (7) is negative definite so that

\[
(A p_i, p_i) < (\lambda_n + \epsilon)(p_i, p_i), \ i = 1, 2, \ldots, r
\]

and

\[
(A p_i, p_k) = (\lambda_n + \epsilon)(p_i, p_k), \ i \neq k; \ i, k = 1, 2, \ldots, r.
\]

Since every \( u \in V_r \) can be written as \( u = \sum_{i=1}^{r} \gamma_i p_i \) we have

\[
R(u) = \frac{\sum_{i=1}^{r} \gamma_i^2 (A p_i, p_i) + \sum_{i \neq k} \gamma_i \gamma_k (A p_i, p_k)}{\sum_{i, k=1}^{r} \gamma_i \gamma_k (p_i, p_k)}.
\]

Using (8) and (9) in (10) we get for every \( u \in V_r \), \( R(u) < \lambda_n + \epsilon \). Combining this with (4) we have \( \lambda_n \leq \max_{u \in V_r} R(u) \leq \lambda_n + \epsilon \). Since \( \epsilon \) can be chosen arbitrarily small the equality (6) holds.

3. Example. We now give an example in which (6) holds for a non-classical choice of \( V_r \). Let \( \lambda_1 < \lambda_2 < \lambda_3 \) and let \( m = r = n = N = 2 \). We choose \( p_1 = u_2 \) and \( p_2 = u_1 + \beta u_3 \) as a basis for \( V_2 \) where \( 0 < \beta^2 \leq (\lambda_3 - \lambda_1)/(\lambda_3 - \lambda_2) \). A simple calculation shows that for every \( u \in V_2 \) the inequality \( R(u) \leq \lambda_2 \) is satisfied. Since \( R(u_2) = \lambda_2 \) we have \( \lambda_2 = \max_{u \in V_2} R(u) \). In this case (7) is a diagonal matrix with elements \(-\epsilon, -\epsilon(1 + \beta^2)\), which verifies our criterion. Let us note the formal analogy to the new maximum-minimum theory of A. Weinstein, where the quantities \((\lambda_j - \lambda)^{-1}, \lambda_n - \epsilon, \) and \( \beta^{-1} \) appear in place of \( \lambda_j - \lambda, \lambda_n + \epsilon, \) and \( \beta \).

4. Concluding remark. It has been shown in [1] and [2],[3] that the roots \( \lambda'_i \leq \lambda'_2 \leq \cdots \leq \lambda'_r \) of (1) satisfy the inequalities

\[
\lambda_1 \leq \lambda'_1, \ \lambda_2 \leq \lambda'_2, \ \cdots, \ \lambda_r \leq \lambda'_r
\]

and that the simultaneous equalities

\[
(11) \quad \lambda_1 = \lambda'_1, \ \lambda_2 = \lambda'_2, \ \cdots, \ \lambda_r = \lambda'_r
\]

are obtained by choosing \( p_k = u_k, \ k = 1, 2, \ldots, r \). In another paper we shall prove that the only \( V_r \) for which (11) holds are those subspaces generated by eigenvectors belonging to \( \lambda_1, \lambda_2, \ldots, \lambda_r \).

References


INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS,
UNIVERSITY OF MARYLAND