TWO THEOREMS IN GEOMETRIC
MEASURE THEORY

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The following two propositions give some new information about
the structure of differentiable maps. We use the symbols \( \mathbb{R}^m \) and \( H^s \)
to designate \( m \) dimensional Euclidean space and \( s \) dimensional Hausdorff measure, respectively.

**THEOREM 1.** If \( m > r \geq 0 \) and \( k \geq 1 \) are integers, \( Y \) is a normed real
vectorspace, \( f: \mathbb{R}^m \to Y \) is \( k \) times continuously differentiable, and

\[
S = \mathbb{R}^m \cap \{ x : \text{dim im } Df(x) \leq r \},
\]

then

\[
H^{r+(m-r)/k}[f(S)] = 0.
\]

**THEOREM 2.** If \( f: \mathbb{R}^m \to \mathbb{R}^n \) is Lipschitzian, \( r \) is an integer, \( 0 \leq r \leq m \),
and

\[
T = \mathbb{R}^n \cap \{ y : H^{m-r}(f^{-1}\{y\}) > 0 \},
\]

then \( H^r \) almost all of \( T \) can be covered by a countable family of \( r \) di­
mensional submanifolds of class 1 of \( \mathbb{R}^n \).

The first theorem optimally sharpens the results of [4], where the
history of the problem is discussed; its proof uses a refinement of the
key lemma in [3], which dealt with the case \( r = 0 \). The second theo­
rem is related to the coarea formulae obtained in [2] and [1]. Proofs
of both theorems will appear in the author's book *Geometric measure
theory*.

**REFERENCES**


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