AHLFORS' CONJECTURE CONCERNING EXTREME SARIO OPERATORS

BY NEVIN SAVAGE

Communicated by A. E. Taylor, March 11, 1966

The linear operators, introduced by Sario \cite{4} to construct harmonic functions with prescribed properties on Riemann surfaces, form a convex set. Ahlfors \cite{1} has conjectured a representation for the extreme operators of this convex set. We give an equivalent formulation of this conjecture and show that it is not true in general.

The author would like to take this opportunity to express his appreciation to Professors H. L. Royden and R. R. Phelps for the numerous conversations on matters relating to this paper.

1. Let $W$ be a subregion of a Riemann surface $R$. We suppose $W$ has a compact complement and that its relative boundary $\alpha$ is analytic. We consider a linear operator $T$ which, to continuous values $f$ on $\alpha$, assigns a harmonic function $Tf$ on $W$ such that $Tf = f$ on $\alpha$. $T$ is assumed to have the following additional properties:

\begin{align*}
(1.1) & \quad T1 = 1, \quad Tf \geq 0 \text{ if } f \geq 0, \\
(1.2) & \quad \int_\alpha \frac{\partial Tf}{\partial n} \, ds = 0.
\end{align*}

Sario \cite{4} has called these operators normal linear operators. It is clear that the set of such operators on $W$ form a convex set.

2. We assume, with Ahlfors \cite{1}, that the ideal boundary $\beta$ of $R$ is analytic. Consider the harmonic measure of the region between $\alpha$ and $\beta$. That is, the harmonic function on $W$ which is 0 on $\alpha$ and 1 on $\beta$ and normalized so that the period of its conjugate function along $\alpha$ is 1. In terms of this conjugate function we parametrize $\alpha$ and $\beta$ by $0 \leq x \leq 1, \ 0 \leq y \leq 1$, respectively.

Given $f$ on $\alpha$, $Tf$ has radial limits almost everywhere on $\beta$ and this null set $E$ may be selected independent of $f$ (See \cite{1}). In this manner we may consider $T$ as inducing a linear mapping from the space $C(0, 1)$ of continuous functions on $\alpha$ to $L^\alpha(0, 1)$ the space of bounded measurable functions on $\beta$. We denote this induced linear operator by $T$ also and the class of all such operators by $L$. They have the following properties corresponding to conditions (1.1) and (1.2) above:

---

\footnote{1 Supported in part by a National Science Foundation Science Faculty Fellowship.}
For a fixed value of $y$ the mapping $f \mapsto (Tf)(y)$ defines a linear functional on $C(0, 1)$. By the Riesz representation theorem we have

\[(2.3) \quad (Tf)(y) = \int_0^1 f(x) \, d\mu(x, y).\]

Corresponding to the above conditions on $T$, the family of measures $\{\mu(\cdot, y)\}$ on $0 \leq x \leq 1$ associated with $T$ has the following properties:

\[(2.4) \quad \mu([0, 1], y) = 1, \quad \mu(\cdot, y) \geq 0\]

\[(2.5) \quad \int_0^1 dy \int_0^1 f(x) \, d\mu(x, y) = \int_0^1 f(x) \, dx.\]

3. Ahlfors [1] has conjectured that $T$ is extreme in $L$ if and only if $\mu(\cdot, y)$ is a point mass for almost every $y$. If this were the case, it is clear that the corresponding operator $T$ is extreme. We note, in this case, that we have a mapping $x = g(y)$ defined for almost every $y$ which assigns to $y$ the point $x = g(y)$ where the point mass is concentrated. Hence $(Tf)(y) = f(g(y))$ and condition (2.5) becomes

\[(3.1) \quad \int_0^1 f(g(y)) \, dy = \int_0^1 f(x) \, dx.\]

Since this holds for every $f$ in $C(0, 1)$, $g(y)$ is a measure preserving transformation from $0 \leq y \leq 1$ to $0 \leq x \leq 1$.

Conversely, if $g(y)$ is any measure preserving transformation from $0 \leq y \leq 1$ to $0 \leq x \leq 1$ then the composition operator defined by $(Tf)(y) = f(g(y))$ is extreme in $L$ and the associated mass distribution $\mu(\cdot, y)$ is a point mass for almost every $y$. Thus an equivalent formulation of Ahlfors' conjecture is that every extreme operator of $L$ is of the form $(Tf)(y) = f(g(y))$ where $g(y)$ is a measure preserving transformation from $0 \leq y \leq 1$ to $0 \leq x \leq 1$.

4. Consider the class $K$ of linear operators which map $C(0, 1)$ into $L^\infty(0, 1)$ and satisfy the conditions:

\[(4.1) \quad T1 = 1, \quad Tf \geq 0 \text{ if } f \geq 0.\]

Then $L = \{T \in K : \int Tf = \int f, f \in C(0, 1)\}$.

$K$ forms a convex set and we determine the extreme operators of $K$. 

AHLFORS' CONJECTURE

\[(2.1) \quad T1 = 1, \quad Tf \geq 0 \text{ if } f \geq 0,\]

\[(2.2) \quad \int_0^1 Tf = \int_0^1 f.\]
Our method is to replace the space $L^\infty(0, 1)$ by an equivalent $C(S)$ space where $S$ is compact Hausdorff and apply the results of Phelps [2] concerning extreme operators which map $C(0, 1)$ into $C(S)$.

The Gel'fand representation (See Dunford and Schwartz, vol. 1, p. 445) gives us an isometric isomorphism $\Lambda$ between $L^\infty(0, 1)$ and $C(S)$ where $S$ is a compact Hausdorff space. The isomorphism $\Lambda$ maps positive functions into positive functions and is an algebraic isomorphism. That is, if $h(y) = f(y)g(y)$ almost everywhere then $\Lambda h = \Lambda f \Lambda g$. Also if $F$ is an arbitrary continuous function of a complex variable and $h$ is in $L^\infty(0, 1)$ then

\begin{equation}
\Lambda(F(h)) = F(\Lambda(h)).
\end{equation}

Note that under the mapping $\Lambda$ a positive function has a positive function as a preimage. For if $\Lambda(f) = g$ where $g \geq 0$ then taking $F$ in (4.2) to be the absolute value function we have $\Lambda(|f|) = |\Lambda(f)| = |g| = \Lambda(f)$. Therefore $f = |f|$ or $f \geq 0$.

Set $T'f = \Lambda(Tf)$. Then $T'$ is a linear mapping from $C(0, 1)$ to $C(S)$. Denote by $K'$ the image of the class $K$ under the map $\Lambda$, then the operators $T'$ in $K'$ satisfy:

\begin{equation}
T'1 = 1, \quad T'f \geq 0 \text{ if } f \geq 0.
\end{equation}

Moreover, $T$ is extreme in $K$ if and only if $T'$ is extreme in $K'$.

**Theorem.** $T$ is extreme in $K$ if and only if there is a bounded measurable mapping $g$ from $0 \leq y \leq 1$ to $0 \leq x \leq 1$ such that $Tf = f(g)$.

**Proof.** By Phelps’ theorem [2], $T'$ in $K'$ is extreme if and only if there is a continuous mapping $G$ from $S$ to $0 \leq x \leq 1$ such that $T'f = f(G)$ for all $f$ in $C(0, 1)$. But $\Lambda(Tf) = T'f = f(G)$ and since $G$ is in $C(S)$ let $g$ be in $L^\infty(0, 1)$ such that $\Delta g = G$. Note that $g \geq 0$ since $G \geq 0$. Also since $\Delta$ is norm preserving we have $\|g\| = \|G\| \leq 1$. Hence $0 \leq g \leq 1$ and thus $g$ is a bounded measurable mapping from $0 \leq y \leq 1$ to $0 \leq x \leq 1$. From (4.2) we have $\Delta(\Lambda(g)) = \Lambda(g)$ for every $f$ in $C(0, 1)$ hence $\Lambda(Tf) = f(G) = \Lambda(g)$ or $Tf = f(g)$.

**Remark.** Phelps [2] has also shown that the extreme operators of $K'$ can be characterized as the multiplicative operators. That is, $T'(fh) = (T'f)(T'h)$. Hence from $\Delta(TfTh) = \Lambda(Tf)\Delta(Th) = (T'f)(T'h) = T'(fh) = \Lambda(T(fh))$ we conclude that

**Theorem.** $T$ is extreme in $K$ if and only if $T$ is multiplicative. That is, $T(fh) = TfTh$ for every $f, h$ in $C(0, 1)$.

5. We now show the connection between Ahlfors' conjecture and
the extreme operators of $L$ and $K$. As noted earlier, Ahlfors' conjecture may be stated as:

$T$ is extreme in $L$ if and only if $Tf = f(g)$ where $g$ is a measure preserving mapping from $0 \leq y \leq 1$ to $0 \leq x \leq 1$.

**Theorem.** Ahlfors' conjecture holds if and only if every $T$ which is extreme in $L$ is also extreme in $K$.

**Proof.** If the conjecture holds then it is clear that $T$ is multiplicative and hence extreme in $K$. Conversely if $T$ extreme in $L$ implies $T$ extreme in $K$ then $Tf = f(g)$ and the integrability condition (2.2) implies that $g$ is measure preserving.

An equivalent formulation of the above result is

**Theorem.** Ahlfors' conjecture holds if and only if every extreme operator $T$ in $L$ is multiplicative.

6. An example due to Ryff [3] shows that there exists an operator $T$ in $L$ which is extreme but not multiplicative. This example settles Ahlfors' conjecture in the negative.

Define the operator $T$ as follows.

\[(Tf)(y) = \frac{f(y/2) + f((y + 1)/2)}{2}.\]

Simple calculations show that $T$ is in $L$ and is not multiplicative. Note that

\[(Tf)(y) = \int_{0}^{1} f(x) d\omega(x, y) = \frac{[f(y/2) + f((y + 1)/2)]}{2}\]

implies that the associated measure $\omega(\cdot, y)$ consists of a 1/2 unit point mass at $y/2$ and a 1/2 unit point mass at $(y + 1)/2$.

To show that $T$ is extreme in $L$, we show that if $T$ can be represented as $T = (T_1 + T_2)/2$ with $T_1$, $T_2$ in $L$, then $T = T_1 = T_2$. In terms of the associated measures we have

\[\omega(\cdot, y) = \frac{[\mu(\cdot, y) + v(\cdot, y)]}{2}\]

where $\mu$ and $v$ are the measures corresponding to $T_1$ and $T_2$, respectively.

Since $\mu$ and $v$ are nonnegative it is evident that each has mass concentrated only at the points $y/2$ and $(y + 1)/2$ for almost every $y$. Also, from the fact that $\omega((y + 1)/2, y) = \omega(y/2, y) = 1/2$ for a.e. $y$ we have

\[1 = \mu(y/2, y) + v(y/2, y)\]

and
(6.5) \[ 1 = \mu((y + 1)/2, y) + \nu((y + 1)/2, y). \]

Let \( \alpha(y) = \mu(y/2, y) \) and \( \beta(y) = \mu((y+1)/2, y) \) then since the total mass of \( \mu \) is 1 we have \( \beta(y) = 1 - \alpha(y) \) for a.e. \( y \). We show that \( \alpha(y) = 1/2 \) for a.e. \( y \) and hence that \( \mu(\cdot, y) = \nu(\cdot, y) \) and thus \( T = T_1 = T_2 \).

From the nature of the mass distribution \( \mu(\cdot, y) \) we have

(6.6) \[ \int_0^1 f(x) \, d\mu(x, y) = \alpha(y)f(y/2) + [1 - \alpha(y)]f((y + 1)/2) \]

for every \( f \) in \( C(0, 1) \). From the integrability condition (2.5) we obtain

(6.7) \[ \int_0^1 f(x) \, dx = \int_0^1 \{ \alpha(y)f(y/2) + [1 - \alpha(y)]f((y + 1)/2) \} \, dy. \]

Let \( f \) in \( C(0, 1) \) be such that \( f \) vanishes on \([1/2, 1]\) then (6.7) becomes

(6.8) \[ \int_0^1 f(x) \, dx = \int_0^1 \alpha(y)f(y/2) \, dy \]

or making a simple change of variable (6.8) is equivalent to

(6.9) \[ \int_0^{1/2} [2\alpha(2t) - 1]f(t) \, dt = 0. \]

This holds for every continuous \( f \) on \([0, 1/2]\) which vanishes at the endpoints. Hence \( \alpha(t) = 1/2 \) for a.e. \( t \) on \([0, 1]\).

REFERENCES


ARIZONA STATE UNIVERSITY