AN ENTIRE TRANSCENDENTAL FUNCTION WHOSE INVERSE TAKES SETS OF FINITE MEASURE INTO SETS OF FINITE MEASURE

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In a recent issue of this journal [2] the following research problem was posed by F. Gross:

Let \( S \) be an arbitrary region of finite measure.\(^1\) Does there exist a transcendental meromorphic function with the property that the pre-image \( f^{-1}(S) \) is of finite measure?

The following theorem answers the above question not just for meromorphic functions but for entire functions also.

**Theorem.** There exists an entire transcendental function \( f(z) \) whose inverse takes sets of finite measure into sets of finite measure.

**Proof.** Let \( D = \{ z = x + iy \mid x > 0, \ |y| < 1/(1+x^2) \} \) and let \( E \) be the complement in the plane of the closure of \( D \). Also let \( c(z) \) be a conformal map from the outside of \( |z - 1| = \epsilon \) (\( \epsilon \) a small positive constant to be determined later) into the unit disk which takes the point at infinity into the origin. By a particular case of Runge’s theorem [3] and Rouche’s theorem there exists a rational function \( R_1(z) \) with the following properties:

(i) \( R_1(z) \) has a pole only at \( z = 2 \).

(ii) \( R_1(z) \) so closely approximates \( c(z) \) on \( E \) that \( R_1(z) \) on \( E \) has the following properties:

(a) measure \( R_1(E) \) less than one.

(b) \( R_1(z) + z \) is 1-1 (this is always possible by choosing \( \epsilon \) small enough) on \( E \).

(c) \( \lambda = |R_1(-1) - R(-2)| \neq 0 \).

By the same argument there exists a rational function \( R_2(z) \) which has the following properties:

(i) \( R_2(z) \) has a pole only at \( z = 3 \).

(ii) \( R_2(z) \) so closely approximates \( R_1(z) \) in \( E \cup G_3(= \{ z = x + iy \mid x < 3/2 \}) \) that \( R_2(z) \) on \( E \cup G_3 \) has the following properties:

(a) measure \( R_2(E) \) less than one.

(b) \( R_2(z) + z \) is 1-1 on \( E \).

(c) \( |R_2(z) - R_1(z)| < \lambda/8 \) on \( E \cup G_3 \).

Continuing inductively there exists a rational function \( R_n(z) \) which has the following properties:

\(^1\) Dr. Gross was kind enough to point out to me that in the statement of the problem accidentally the words “of finite measure” were left out.
(i) $R_n(z)$ has a pole only at $z = n + 1$.

(ii) $R_n(z)$ so closely approximates $R_{n-1}(z)$ in $E \cup G_{2n-1}$ ($= \{ z = x + iy \mid x < (2n - 1)/2 \}$) that $R_n(z)$ on $E \cup G_{2n-1}$ has the following properties:

(a) measure $R_n(E)$ less than one.

(b) $R_n(z) + z$ is 1-1 on $E$.

(c) $|R_n(z) - R_{n-1}(z)| < \lambda/2^{n+1}$ on $E \cup G_{2n-1}$.

By a theorem of Ostrowski [1] on the convergence of meromorphic functions in the chordal metric $\lim_{n \to \infty} R_n(z) = h(z)$ exists and is meromorphic in the plane. However by the way we choose the poles of $R_n(z)$ to converge to infinity $h(z)$ is clearly analytic. Since measure of $h(E)$ is less than or equal to one $h(z)$ cannot be a nonconstant polynomial and $h(z)$ cannot be constant since $|h(-1) - h(-2)| \geq \lambda/2$.

Now let $f(z) = h(z) + z$ and let $S$ be any set of finite measure in the plane. The part of $f^{-1}(S)$ contained in $D$ certainly has finite measure. The part of $f^{-1}(S)$ in $E$ (call it $K$) must have finite measure, otherwise we would be led to a contradiction since $\text{[measure of } f(K)\text{]} \geq \text{[measure of } K\text{]} - \text{[measure of } h(K)\text{]}$ (from $|f'(z)|^2 \geq 1 - |h'(z)|^2$ for all $z$ in $E$ and the fact $f(z)$ has to be 1-1 on $E$) and the measure of $h(K)$ is less than one.

References

