COBORDISM OF GROUP ACTIONS

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Let $G$ be a compact Lie group and $M$ a compact $G$ manifold without boundary, i.e. a $C^\infty$ manifold with a differentiable action of $G$ on $M$. $M^n$ is said to be $G$-cobordant to zero $M \sim_0 0$ if there exists a compact $G$ manifold $Q^{n+1}$ with $\partial Q = M$. Note that in this case $M_\sigma$ (the fixed point set of $M$) = $\partial Q_\sigma$. $M_\sigma$ and $Q_\sigma$ are both disjoint unions of closed submanifolds (of varying dimension) of $M$, $Q$ respectively.

Let $\nu(M, Q)$ denote the normal bundle of $M_\sigma$ in $M$; $\nu(M_\sigma, M) \to M_\sigma$ is a $G$-vector bundle in the sense of [5]. A partial converse to the statement $\nu(M_\sigma, M) = \partial Q_\sigma$ is given by

**PROPOSITION 1 ([2, p. 10]).** If $\nu(M_\sigma, M)$ is cobordant to zero as a $G$-vector bundle, i.e. if there exists a manifold $W$ and a $G$-vector bundle $E \to W$ with $\partial W = M_\sigma$, $E | \partial W = \nu(M_\sigma, M)$ then $M$ is $G$-cobordant to a manifold $M'$ with $M'_\sigma \equiv \emptyset$.

**PROOF.** Form the manifold $M \times I \cup_f E(1)$ where $E(1)$ denotes the unit disc bundle in $E$ and

$$f: E(1) \mid \partial W = \nu(M_\sigma, M) \xrightarrow{\exp_{\text{fix}}} M \times 1.$$ 

Then note that, after smoothing,

$$\partial(M \times I \cup_f E(1)) = M \times 0 \cup (M \times 1 - f(E(1) \mid \partial W)) \cup \partial E(1)$$

$$= M \times 0 \cup M'.$$

Hence, one may view the $G$-cobordism class of $\nu(M_\sigma, M)$ as a first obstruction to finding a cobordism $M \sim_0 0$. Higher obstructions are formulated in terms of a spectral sequence. For simplicity we deal only with the unoriented case.

Let $V$ be an orthogonal representation of $G$ and let $V^n$ denote the $n$-fold direct sum of $V$ with itself and $S(V)$ the unit sphere in $V$. Consider the category of manifolds $\mathcal{S}(V)$ where $M$ is in $\mathcal{S}(V)$ iff $M$ can be imbedded in $S(V^n)$ for some $n$. One can then define the cobordism groups $\pi_n(V) = \pi_n(\mathcal{S}(V))$ of $n$ dimensional $G$-manifolds in $\mathcal{S}(V)$ (see [5]). It was shown in [5] that if $G$ is finite or abelian then $\pi_n(V) \cong \pi_1^{g_{2n+3}}(T_k(V^{2n+3} \oplus R), \infty)$ where $\pi_1^{g_{2n+3}}(T_k(V^{2n+3} \oplus R), \infty)$ de-

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notes the equivariant homotopy classes of maps of \( S(V^{2n+3} \oplus R) \) into \( T_k(V^{2n+3} \oplus R) \) the Thom space of the universal bundle of \( k \)-planes in \( V^{2n+3} \oplus R \). Let \( f \) be such a map; then proposition 1 may be reinterpreted as saying

**Proposition 1'.** Any homotopy of

\[
f | S(V^{2n+3} \oplus R)_G : S(V^{2n+3} \oplus R)_G \to T_k(V^{2n+3} \oplus R)_G
\]

may be covered by a homotopy of \( f \).

It was shown in [5] that there are only a finite number of conjugacy classes of isotropy groups occurring in \( G(V) \); let \( (H_1), \cdots, (H_r) \) denote the conjugacy classes ordered by \( (H_i) < (H_j) \) iff there is a \( g \in G \) with \( gH_ig^{-1} \subset H_j \) but \( gH_ig^{-1} \neq H_j \). Define the level \( H_i > n \) if \( H_i < H_j \) and level \( H_i > n - 1 \); level \( G = 0 \) by definition and level \( H_i = n \) if level \( H_i > n - 1 \) but not level \( H_i > n \). We may filter \( G(V) \) by subcategories \( G^i(V) \) where \( M \) is in \( G^i(V) \) if for each \( x \in M \) level \((G_x) \geq i \) and \( G_x \) is the isotropy group of \( x \). One then has the corresponding cobordism groups \( D_{n,i} = \Xi_{n_i}(G^i(V)) \). Let \( D_{n,0} = D_{n,0} \) for \( i \leq 0 \) and let \( D^{n,i} \) denote the image of \( D_{n,i} \) in \( D_{n,0} = \Xi_{n_i}(V) \). We define \( E_{n,i} = 0 \) for \( i < 0 \). Vector bundles with fibre dimension zero are included.

**Theorem.** There is a graded exact couple

\[
\begin{align*}
D & \xrightarrow{\partial} D \\
\partial & \downarrow _\nu \quad \Downarrow \\
E & \xrightarrow{\nu}
\end{align*}
\]

where

\[
D = \sum_{n,i} D_{n,i}, \quad E = \sum_{n,i} E_{n,i}
\]

\( \text{with} \)

\[
E_{n,i} = E_{n,i} = D^{n,i} / D^{n,i+1}.
\]
In particular

\[ \mathfrak{N}_n(V) \approx \sum_{i=0}^{\infty} E_{n,i}^\infty. \]

The maps are as follows: Define \( w: D_{n,i} \to D_{n,i-1} \) by \( w([M]) = [M] \); if \( M \) is in \( \mathcal{G}^i(V) \) then \( M \) is in \( \mathcal{G}^{i-1}(V) \). Define \( \partial: E_{n,i} \to D_{n-1,i+1} \) by \( \partial(E \to M) = S(E) \); \( \partial \) is well defined by (i), (ii), and (iii). Define \( \nu: D_{n,i} \to E_{n,i} \) by \( \nu([M]) = [\nu(M_i, M)] \) where \( M_i = \{ x \in M \mid \text{level } (G_x) = i \} \); \( M_i \) is a closed submanifold since \( M \) is in \( \mathcal{G}^i(V) \). Conditions (i)–(iv) are clearly satisfied. Exactness follows from straightforward geometric arguments.

The groups \( E_{n,i} \) may be described as follows: let \( H \) be an isotropy group on level \( i \) and let \( W \) be an \( r \) dimensional representation of \( H \) with \( W \subseteq V^*|H \) for some \( s \) where \( V^*|H \) means \( V^* \) considered as an \( H \) space.

Let \( P(H, W) \) be the group of \( N(H) \) (normalizer of \( H \) in \( G \)) equivariant bundle maps of \( W \times H \) into itself which are diffeomorphisms on the base space \( N(H)/H \). We have the exact sequence \( 0 \to O_H(W) \to P(H, W) \to N(H)/H \to 0 \) where \( O_H(W) \) is the group of \( H \) equivariant orthogonal transformations of \( W \).

**Proposition 2.** \( E_{n,i} \) is isomorphic to the direct sum of \( \mathfrak{N}_i(BP(H, W)) \) over all such representations of \( H \) and all conjugacy classes of subgroups on level \( i \); \( \mathfrak{N}_i(BP(H, W)) \) denotes the ordinary cobordism group (see [1, p. 45]) of the classifying space of \( P(H, W) \) and \( t = n - \dim W - \dim G/H \).

**Proof.** Let \( E \to M \) be a bundle in \( E_{n,i} \) with \( (G_x) = (H) \) for all \( x \in M \). By equivariance, it suffices to consider the \( N(H) \) bundle \( E \mid M_H \to M_H(M_H = \{ x \in M \mid G_x = H \}) \) since \( M = M_H \times_{N(H)} G \) ([3, p. 42]); but \( M_H \) is a \( N(H)/H \) principal bundle over \( M/G \) and hence one can see that \( E \mid M_H \to M_H \to M/G \) is an \( N(H) \) fibre bundle with fibre \( N(H) \times_H W \) and structural group \( P(H, W) \) ([4, p. 40]). Any element of \( E_{n,i} \) is the disjoint union of such bundles.

To describe the differential we let \( K \subseteq H \) be an isotropy group on level \( i+1 \); then \( W \mid K = W_0 \oplus W_1 \) where \( K \) operates trivially on \( W_0 \). \( S(W_0) \) is a \( N(K, H)/K \) principal bundle where \( N(K, H) \) denotes the normalizer of \( K \) in \( H \). Form the \( N(H) \) bundle \( U \) over \( BP(H, W) \) with fibre \( N(H) \times_H S(W) \), \( U = E_P \times_{P} (N(H) \times_H S(W)) \) where \( E_P \to BP(H, W) \) is the universal principal bundle; then \( U(K)/N(H) = U(K) \) is a bundle over \( BP(H, W) \) with fibre \( S(W_0)/N(K, H) \) and there is a map \( i: U(K) \to BP(K, W_1) \) which classifies the normal
bundle of $U(K)$ in $U$. Then for any $[M, f] \in \Xi_{i}(BP(H, W))$ we have the diagram

$$
\begin{array}{ccc}
\pi_{*}U(K) & \xrightarrow{f*} & U(K) \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & BP(H, W).
\end{array}
$$

Clearly $d([M, f]) = \sum [f*U(K), \iota \circ f_*] \in E_{n-1,i+1}$ where $[f*U(K), \iota \circ f_*] \in \Xi_{s}(BP(K, W_1)$, $s=n-1-\dim W_1-\dim G/K$, and the sum extends over all conjugacy classes $(K)$ on level $i+1$ with $K \subset H$.

**References**


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