Consider the following two conjectures:

\( C(n) \): (The combinatorial Schoenflies conjecture.) A combinatorial \((n - 1)\)-sphere on a combinatorial \(n\)-sphere decomposes the latter into two combinatorial \(n\)-cells.

\( D(n) \): Let \( W^n \) be an orientable combinatorial manifold without boundary and let \( M^{n-1} \) be a closed orientable combinatorial manifold imbedded piecewise linearly in \( W^n \). Let \( U \) be a regular neighborhood of \( M^{n-1} \) in \( W^n \). Then there exists a piecewise linear homeomorphism \( h: M^{n-1} \times [-1; 1] \rightarrow U \) such that

\( h(x, 0) = x, \)

\( h \) is onto.

It is easily seen that \( D(n) \) implies \( C(n) \) for all \( n \neq 4 \) by using the Hauptvermutung for combinatorial cells and spheres [10]. In [8], Noguchi shows that \( C(1), C(2), \ldots, C(n) \) imply \( D(n+1) \). By using the fact that a compact component of the boundary of a combinatorial manifold is combinatorially collared [9], [11], it is easily shown that \( C(n) \) implies \( D(n+1) \). However it is possible to prove a weaker version of \( D(n+1) \) without the use of \( C(n) \) for the special case when \( W, M \) are spheres.

**Theorem.** Let \( \sum^n \) \((n \neq 4) \) be a combinatorial sphere embedded piecewise linearly in the combinatorial sphere \( S^{n+1} \). Let \( U \) be a regular neighborhood of \( \sum^n \) in \( S^{n+1} \). Then there exists a piecewise linear homeomorphism \( h: \sum^n \times [-1; 1] \rightarrow S^{n+1} \) such that \( h(\sum^n \times [-1; 1]) = U \).

**Proof.** (For definitions of terms used see [11].) Since \( C(i), i = 1, 2, 3, \) is valid [1], [6], it follows from the remarks above that the theorem is true for \( n < 4 \). Suppose \( n > 4 \).

Since \( \sum^n \) is a deformation retract of \( U \), the \( i \)th integral homology groups of \( \sum^n \) and \( U \) are isomorphic for all \( i \). It follows then from Alexander duality and the unicoherence of the sphere that the closure of \( S^{n+1} - U \), \( \text{Cl}(S^{n+1} - U) \), is the union of two connected closed sets,

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$D_1, D_2$ with a connected boundary $T_1, T_2$ respectively. Since $U$ is a combinatorial manifold, from [2], we have that each $D_i$ is a combinatorial manifold. Similarly, $S^{n+1} - \sum^n = R_1 \cup R_2$ where $D_i \subset R_i$ and $\text{Cl} \, R_1 \cap \text{Cl} \, R_2 = \sum^n$. By either [3] or [7], $\text{Cl} \, R_1$ and $\text{Cl} \, R_2$ are topological $(n+1)$-cells.

We want to show that each $T_i$ is simply connected. Let $f: S^1 \to T_1$ be a continuous map of the 1-sphere into $T_1$. By the simplicial approximation theorem, we may assume $f$ is piecewise linear. Since $U$ is simply connected (for it is of the same homotopy type as $\sum^n$), $f(S^1)$ bounds a disk $N$ in $U$. We may assume $N$ is polyhedral and in general position with respect to $\sum^n$. Then if $N \cap \sum^n \neq \emptyset$, $N \cap \sum^n$ is a finite collection of simple closed curves. Since $\sum^n$ is simply connected, we can suppose that $N$ lies in $U \cap \text{Cl} \, R_1$; for by the usual alteration techniques, see, for example, [4], we can replace $N$ by a disk which is bounded by $f(S^1)$ and lies in $U \cap \text{Cl} \, R_1$. By using the collar of the boundary of $\text{Cl} \, R_i$, we can assume that $N \cap \sum^n = (T_1 \cup T_2) \times [0, 1)$. Since $U - \sum^n = (T_1 \cup T_2) \times [0, 1)$, we can then push $N$ into $T_1$.

Since $D_i \cup U \setminus \text{Cl} \, R_i$, $i = 1, 2$, it follows that each $D_i \cup U$ is contractible and hence from the fact that each $T_i$ is bicollared and from duality, each $D_i$ has homology groups of a point. Since each $T_i$ is simply connected it follows from a similar argument as above that each $D_i$ is simply-connected. From the Hurewicz Isomorphism Theorem, it follows that each $D_i$ is contractible. Hence from [10], we have that each $D_i$ is a combinatorial $(n+1)$-cell. From [2], each $\text{Cl}(S^{n+1} - D_i)$ is a combinatorial $(n+1)$-cell. Then $U = \text{Cl}(\text{Cl}(S^{n+1} - D_i) - D_2)$ is piecewise linear homeomorphic to $\sum^n \times [-1; 1]$ [11].

REMARKS. Attempts to prove the above theorem for manifolds not spheres by the techniques of Noguchi fail because of the missing dimension $n = 4$. From [5], it follows that $T_1 \times (0; 1)$ is topologically homeomorphic to $S^n \times (0; 1)$, but otherwise it is unknown to the author whether $T_1$ is a topological 4-sphere in the case $n = 4$.

REFERENCES


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