1. Introduction and statement of results. In this note we indicate a method of performing surgery on piecewise linear (= PL) manifolds, and show how to prove piecewise linear analogs of theorems on the homotopy type and classification of smooth manifolds\(^2\) (Browder [1], Novikov [10], Wall [13]).

The basic principles are two: to use normal microbundles instead of normal vector bundles, and to put a differential structure \(\sigma\) on a neighborhood \(V\) of an embedded sphere \(S^d\) that represents a homotopy class we wish to kill. Then smooth ambient surgery can be performed on \(V\), and the resulting cobordism triangulated.

Let \(M_1, M_2\) be closed PL \(n\)-manifolds embedded in \(S^{n+k}\) with normal microbundles \(v_1, v_2\). A normal equivalence \(b: (M_1, v_1) \rightarrow (M_2, v_2)\) is a microbundle equivalence \(b: v_1 \rightarrow v_2\) covering a homotopy equivalence \(M_1 \rightarrow M_2\).

Let \(T(v_i)\) be the Thom complex of \(v_i\) (see [12]), and let \(c_i \in \pi_{n+k}T(v_i)\) be the homotopy class of the collapsing map \(S^{n+k} \rightarrow T(v_i)\). We call \(c_i\) a normal invariant for \(M_i\). If \(\partial M \neq 0\), a similar construction defines a normal invariant for \(M\) as an element in \(\pi_{n+k}(T(v_M), T(v_M|\partial M))\).

**Theorem 1.** Let \(X\) be a 1-connected polyhedron satisfying Poincaré duality in a dimension \(n \geq 5\). Let \(\xi\) be a PL \(k\)-microbundle over \(X\), and let \(\alpha \in \pi_{n+k}T(\xi)\) be such that \(h(\alpha) = \Phi(g)\), where \(h: \pi_{n+k}T(\xi) \rightarrow H_{n+k}T(\xi)\) is the Hurewicz homomorphism, \(\Phi: H_n(X) \rightarrow H_{n+k}T(\xi)\) is the Thom isomorphism, and \(g \in H_n(X)\) is a generator. Assume \(k \geq n\). Then \(X\) has the homotopy type of a closed PL \(n\)-manifold \(M \subset S^{n+k}\) such that

(a) If \(n\) is odd, or if \(n = 4q\) and the signature of \(X\) is \(\langle L_q(\xi_1), \cdots, L_q(\xi_q), g \rangle\), then \(M\) has a normal microbundle induced from \(\xi\), and \(\alpha\) is a normal invariant of \(M\);

(b) If \(n\) is even, \(M - \{\text{point}\}\) has a normal microbundle induced from \(\xi\).

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\(^2\) We are informed that some of our results have been obtained independently by R. Lashof and M. Rothenberg.
Theorem 1 is the PL analog of [1]; see also [10].

**Theorem 2.** Let $M_1, M_2$ be PL closed 1-connected $n$-manifolds $n \geq 5$. Then $M_1$ and $M_2$ are combinatorially equivalent if and only if there are normal microbundles $\nu_i$ ($i = 1, 2$) of embeddings $M_i \subset S^{n+k}$, with normal invariants $c_i \in \pi_{n+k} T(\nu_i)$, and a normal equivalence $b: (M_1, \nu_1) \rightarrow (M_2, \nu_2)$ such that $T(b)_{*}(c_1) = c_2$.

Theorem 2 is the PL analog of a theorem of Novikov [10].

**Corollary.** Let $M$ be a PL closed 1-connected $n$-manifold, $n \geq 5$. Suppose the natural map $k_{\text{PL}}(M) \rightarrow k_{\text{Top}}(M)$ is injective and that $k_{\text{PL}}(\Sigma M) \rightarrow k_{\text{Top}}(\Sigma M)$ is surjective (see [8] and [9]) where $\Sigma M$ is the suspension of $M$. Then the PL structure on the underlying topological manifold $M$ is unique up to isomorphism.

**Proof.** Let $\nu_1, \nu_2$ be normal microbundles of two PL structures $M_1, M_2$ on $M$. By the stable uniqueness of a topological normal microbundle of $M$ [8], and the injectivity of $k_{\text{PL}}(M) \rightarrow k_{\text{Top}}(M)$, it follows that $\nu_1$ and $\nu_2$ are stably equivalent as PL microbundles. Let $c_i \in \pi_{n+k} T(\nu_i)$ be the normal invariant of $M_i$. Since $M_1$ and $M_2$ are the same topological manifold, it follows that (for sufficiently large $k$) there is a topological microbundle equivalence $b: \nu_1 \rightarrow \nu_2$ such that $T(b)_{*}(c_1) = c_2$. (The stable tubular neighborhood theorem [4], [7] is needed.) Using the surjectivity of $k_{\text{PL}}(\Sigma M) \rightarrow k_{\text{Top}}(\Sigma M)$ we can choose $b$ to be a PL microbundle equivalence. The Corollary follows from Theorem 2.

**Theorem 3.** Let $(X, A)$ be a polyhedral pair with both $X$ and $A$ 1-connected, satisfying Poincaré duality in a dimension $n \geq 6$. Let $\xi$ be a PL $k$-microbundle over $X$ with $k > n$, let $e \in H_n(X, A)$ be a generator, and suppose there exists $\beta \in \pi_{n+k}(T(\xi), T(\xi, A))$ such that $h(\beta) = \Phi(e)$. Then $(X, A)$ is homotopy equivalent to PL manifold with boundary $(M, \partial M)$ having a normal microbundle induced from $\xi$, and having $\beta$ for a normal invariant. Moreover, $M$ is unique up to PL homeomorphism.

This is the PL analog of a result of Wall [13].

2. **Proofs of theorems.** We indicate the modification in the proofs of the analogous smooth theorems that are required in the PL case.

To prove Theorem 1, by using the transverse regularity theorem
of Williamson [12] we may assume that there is a PL closed $n$-manifold $N \subset S^{n+k}$ such that:

(i) if $f: S^{n+k} \to T(\xi)$ represents $\alpha$, then $f^{-1}(X) = N$;

(ii) if $f|N = f$, then $f^*\xi = \nu$, the normal microbundle of $N$ in $S^{n+k}$;

(iii) $f: N \to X$ has degree 1.

(See [1].)

**Main Lemma.** Let $S \subset N$ be a PL embedded $p$-sphere, $p < n/2$, such that $f|S: S \to X$ is null homotopic. Then there exists a PL surgery killing the homotopy class of $S$. If $N'$ is the resulting $n$-manifold the trace of the surgery (an elementary PL cobordism $K$ between $N$ and $N'$) can be embedded in $S^{n+k} \times I$ with $K \cap (S^{n+k} \times 0) = N = N \times 0$ and $K \cap (S^{n+k} \times 1) = N'$. Moreover, $K$ has a PL normal microbundle $\eta$ in $S^{n+k} \times I$ with $\eta = g^*\xi$, where $g: K \to X$ extends $f: N \to X$.

**Proof.** Let $U \subset N$ be an open regular neighborhood of $S$. Then $f^*\xi|U = \nu|U$ is trivial because $f|U$ is null homotopic. Therefore there is a PL embedding $\phi: U \times R^k \to S^{n+k}$ such that $\phi(x, 0) = x$ and $\phi^{-1}N = U \times 0$. By the product theorem of [5], the smoothing of $U \times R^k$ induced by $\phi$ is concordant to a product smoothing. In fact, there is an open neighborhood $V$ of $S$ in $N$ with $\bar{V} \subset U$, a smoothing $\sigma$ of $V$, and a piecewise differentiable isotopy $\phi_t: U \times R^k \to S^{n+k}$ such that

(i) $\phi_0 = \phi$,

(ii) $\phi_t = \phi$ outside $V \times R^k$,

(iii) $\phi_t|V \times D^k$ is a smooth embedding $V \times D^k \to S^{n+k}$.

Observe now that $\phi_t(V \times 0)$ is a smooth submanifold of $S^{n+k}$ and $\phi_t$ provides a trivialization of its normal vector bundle. Let $V' \subset V_t$ be a smooth closed neighborhood of $S$, and put $W_0 = \phi_1(V' \times 0)$. Let $W_1 \subset S^{n+k}$ be the smooth submanifold obtained from $W_0$ by a smooth surgery killing the homotopy class of $\phi(S \times 0)$. By Haefliger [2] the trace of the surgery is a cobordism $L$ between $W_0$ and $W_1$ smoothly embedded in $S^{n+k} \times I$ such that $\partial L = W_0 \times 0 \cup (\partial W_0) \times I \cup W_1 \times 1$, and such that the embedding is the product embedding in a neighborhood of $\partial W_0 \times I$. Furthermore, the map $f': W_0 \times 0 \cup (\partial W_0) \times I \to X$, defined to be the composition

$$(W_0 \times 0) \cup (\partial W_0) \times I \to W_0 \phi_1^{-1} \to N \to X$$

extends to $f''': L \to X$ such that $f'''\xi$ is the normal bundle of $L$ in $S^{n+k} \times I$. 

1966] SURREY ON PIECEWISE LINEAR MANIFOLDS 961
The cobordism $L$ and the product cobordism $(N - \text{int } V') \times I$ fit together to form a cobordism $K_1 \subset S^{n+k} \times I$ between $N \times 0$ and $((N - \text{int } V') \cup W_1) \times 1$. The composition

$$(N - \text{int } V') \times I \to N \to X$$

and $f'' : L \times X$ fit together to give a map $g : K_1 \to X$. The microbundle $\nu$ extends to a microbundle $\eta$ over $K_1$ that coincides with $\nu$ over $N \times 0$, with $\nu \times I$ over $(N - \text{int } V') \times I$, and such that $\phi_1$ is a trivialization of $\eta|_{W_1 \times 1}$. In fact, $\eta = g^* \xi$. The isotopy $\phi_t$ provides an embedding $G : E \eta \to S^{n+k} \times I$ of the total space $\eta$ which is the identity on $E \nu$. Consider $G$ as a smooth triangulation of an open subset of $S^{n+k} \times I$.

Whitehead's triangulation theorems show that there is a neighborhood $E_0$ of the zero section of $\eta$ and a homeomorphism $H$ of $S^{n+k} \times I$ such that $HG|_{E_0}$ is PL, and $H|_{S^{n+k} \times 0}$ is the identity. Thus $K = HG(K_1)$ is the desired cobordism. This completes the proof of the Main Lemma.

The proof of Theorem 1 proceeds as in the smooth case if $n$ is odd. If $n$ is even, we proceed until we have an $N$ such that $f : N \to X$ is an isomorphism in homotopy below the middle dimension. Following the procedure of the proof of the main lemma, we find just as in the smooth case that the obstruction $c$ to surgery is a signature or Kervaire-Arf invariant of the intersection quadratic form on the kernel $K_r$ of $f_*|_{H_r(N)}$ in $H_r(N; \mathbb{Z}_2)$. If the signature of $X$ is as in (a) of Theorem 1, then $c = 0$; otherwise $c \equiv 0 \mod 8$. (To see this, recall that a nonsingular quadratic form taking only even values has signature divisible by 8. It suffices to prove $x \# x = 0$ for $x \in \ker(f_*|_{H_r(N; \mathbb{Z}_2)})$. If $p : H^*(N; \mathbb{Z}_2) \to H_*(N; \mathbb{Z}_2)$ is Poincaré duality, then $x \# y = \langle p^{-1}x \cup p^{-1}y, N \rangle$ for $x, y \in H_*(N; \mathbb{Z}_2)$. Let $p^{-1}x = z$. Then $x \# x = \langle Sq z, N \rangle = \langle x \cup U_N, N \rangle$ where $U_N \in H^*(N)$ is the total Wu class. Since $Sq^{-1}U_N = W_N$ (the total Stiefel-Whitney class of $N$), if we define $U_x = Sq^{-1}W(\xi)^{-1}$ it follows that $U_N = f^*U_x$, and $x \# x = \langle Z \cup f^*U_x, N \rangle = x \# Pf^*U_x$. By [1], $K_r$ is orthogonal to $Pf^*(H^*(X))$. Hence $x \# x = 0$.)

There exists an oriented PL closed $(r - 1)$-connected $2r$-manifold $P$ with signature $-8$ if $r = 2q$, and with Kervaire-Arf invariant 1 if $r = 2q + 1$. Moreover $P-\{\text{point}\}$ is parallelizable smoothable. It follows [3] that there is a PL embedding $P \subset S^{2r+2} \times I$ having a trivial normal bundle on $P_0$ (the complement of a highest dimensional cell). Therefore the connected sum $N \# P$ embeds in $S^{n+k}$ with a normal microbundle $\nu'$ on $(N \# P)_0$ which coincides with the normal microbundle $\nu$ of $N$ on $N_0$, and which is trivial on the rest of $(N \# P)_0$. 


Let $N' = N \# P$ if $r = 2g + 1$, and let $N'$ be the connected sum of $N$ with $c/8$ copies of $P$ if $r = 2g$. Define $f': N' \to X$ by $f'|N_0 = f$, and $f|N' - N_0$ constant. Since $\nu'|N' - N_0$ is trivial, $f'$ is covered by a microbundle map $\nu' \to \xi$. The obstruction to surgery on $N'$ now vanishes. Hence by surgery we obtain a manifold $M \subset S^{n+k}$ with a normal microbundle $\nu$ on $M_0$ and a homotopy equivalence $f: M \to X$ such that $f|M_0$ is covered by a microbundle map $\nu \to \xi$.

Alternatively, in the middle dimension we could use the method of [14].

Theorem 2 is proved in a similar way, using the same trick to extend Novikov’s proof to the PL case. Since for $n \geq 5$ any PL homotopy sphere $T$ is a combinatorial sphere (Smale [11]), the conclusion of the smooth case, that $M_1 \# T = M_2$ becomes $M_1 = M_2$ in the PL case.

For Theorem 3 we imitate the proof of Theorem 2 of Wall [13] with the following modification of the immersion argument of [13]. Given a PL map $f: D^{k+1} \to M^{2k+1}$ (in the notation of [13]), assume that $f$ has generic singularities. It follows that $H^i(f(D^{k+1})) = 0$ for $i > 2$. Since $\Gamma_i = 0$ for $i \leq 2$, it follows from [5] that a neighborhood $V$ of $f(D^{k+1})$ in $M^{2k+1}$ has a smoothing $\sigma$. Then we approximate $f$ by a smooth map into $V_\sigma$ and proceed as in [13].

**Bibliography**


PRINCETON UNIVERSITY AND
UNIVERSITY OF CALIFORNIA, BERKELEY