

# ON THE DECOMPOSITION OF INVARIANT SUBSPACES

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**1. Introduction.** The main result of this paper is a decomposition theorem for invariant subspaces of operators on Banach spaces. The theorem is closely related to a decomposition theorem of F. Riesz.

We apply the theorem to the study of invariant subspaces of direct sums of operators, producing some new examples of lattices of invariant subspaces of operators on Hilbert space.

**2. The Main Theorem.** Let  $B$  be a (complex) Banach space and  $T$  a bounded linear operator on  $B$ . We use  $\sigma(T)$  to denote the spectrum of  $T$ . Suppose that  $\sigma(T) = \sigma_1 \cup \sigma_2$ , where  $\sigma_1$  and  $\sigma_2$  are disjoint non-empty closed sets. Then, as Riesz has shown, [4, §148], one can choose contours  $\gamma_j$ ,  $j = 1, 2$ , with  $\gamma_j$  surrounding  $\sigma_j$ , such that if

$$P_j = -\frac{1}{2\pi i} \int_{\gamma_j} (T - z)^{-1} dz,$$

then  $P_j$  is a projection onto an invariant subspace  $B_j$  of  $T$ . Moreover  $B = B_1 \vee B_2$ ,  $B_1 \cap B_2 = \{0\}$ , and  $\sigma(T|_{B_j}) = \sigma_j$ .

We strengthen the hypothesis in Riesz's Theorem and get a stronger conclusion. If  $S$  is a compact subset of the complex plane let  $\eta(S)$  denote the union of  $S$  and all "holes" in  $S$ ; i.e.,  $\eta(S)$  is the union of  $S$  and all bounded components of the complement of  $S$ .

**THEOREM 1.** *If  $\eta(\sigma_1) \cap \eta(\sigma_2) = \emptyset$ , then every invariant subspace  $M$  of  $T$  has a unique decomposition of the form  $M = M_1 \vee M_2$ , where  $M_j$  is invariant under  $T$  and  $M_j \subset B_j$ ,  $j = 1, 2$ .*

To prove the theorem we require several lemmas.

**LEMMA 1.** *If  $\eta(\sigma_1) \cap \eta(\sigma_2) = \emptyset$ , then  $\eta(\sigma(T)) = \eta(\sigma_1) \cup \eta(\sigma_2)$ .*

Lemma 1 follows immediately from a property of the extended complex plane (or 2-sphere) closely related to unicoherence. (For a proof of the relevant property see [6, p. 60].)

Now let  $M$  be any nontrivial invariant subspace for  $T$  and let  $R$  be the restriction of  $T$  to  $M$ .

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LEMMA 2.  $\sigma(R) \subset \eta(\sigma(T))$ .

In fact, more is true: if one point of a hole is in  $\sigma(R)$  then the entire hole is in  $\sigma(R)$ .

Lemma 2 was proved by Bram [1] in the case that  $T$  is a normal operator on Hilbert space. S. Parrott has observed (unpublished) that the general result follows directly from the easily established relations

$$\delta(\sigma(R)) \subset \pi(R) \subset \pi(T) \subset \sigma(T),$$

where  $\pi$  denotes approximate point spectrum [3, p. 51], and  $\delta(\sigma(R))$  denotes the boundary of  $\sigma(R)$ .

LEMMA 3. *If  $f$  is a function analytic on an open set containing  $\eta(\sigma(T))$ , then  $f(R) = f(T) | M$ .*

First of all, by Lemma 2,  $f(R)$  is defined. By Riesz's functional calculus, [4, § 151],  $f(T)$  has a representation

$$f(T) = -\frac{1}{2\pi i} \int_{\gamma} f(z)(T - z)^{-1} dz,$$

where  $\gamma$  is a contour surrounding  $\eta(\sigma(T))$ . Lemma 3 now follows from the fact that if  $z \in \gamma$  then  $z \notin \sigma(R)$  and  $z \notin \sigma(T)$ , and thus  $(R - z)^{-1} = (T - z)^{-1} | M$ .

We can now prove Theorem 1. By Lemmas 1 and 2,  $\sigma(R) = \tau_1 \cup \tau_2$ ,  $\tau_j \subset \eta(\sigma_j)$ , (where either of the  $\tau_j$  may be empty). Now if one considers the Riesz projections for  $T$  discussed above, where the contours  $\gamma_j$  are chosen so that  $\gamma_j$  encloses  $\eta(\sigma_j)$  and is disjoint from  $\eta(\sigma(T))$ , Lemma 3 implies that the Riesz projections for  $R$ , (corresponding to  $\tau_1$  and  $\tau_2$ ), are restrictions of the Riesz projections for  $T$ . The theorem follows.

REMARK. In Theorem 1 it does not suffice to assume merely that  $\sigma_1 \cap \sigma_2 = \emptyset$ . A counterexample is provided by  $T = U \oplus 0$  on  $H \oplus H$ , where  $H$  is a separable infinite-dimensional Hilbert space and  $U$  is a bilateral shift on  $H$ . Here  $\sigma_1 = \sigma(U) = \{z : |z| = 1\}$  and  $\sigma_2 = \{0\}$ .

### 3. Application to lattices of invariant subspaces.

DEFINITION. An abstract lattice  $L$  is an  $S$ -lattice ( $S$  for subspace) if  $L$  is order-isomorphic to the lattice of invariant subspaces of some bounded linear operator on a separable complex Hilbert space.

THEOREM 2. *If  $\{L_n\}$  is a finite or countably infinite family of  $S$ -lattices then the direct product  $\times_n L_n$  is an  $S$ -lattice.*

To prove Theorem 2, suppose that  $L_n$  is attained by an operator

$T_n$  on the Hilbert space  $H_n$ . Let  $\{D_n\}$  be a sequence of disjoint closed discs in the complex plane which have radii tending to zero and centers tending to zero along the real axis. By multiplying and translating the  $T_n$ 's by appropriate scalars we can assume that they are uniformly bounded in norm and that  $\sigma(T_n) \subset D_n$  for each  $n$ .

Now let  $T = \sum_n \oplus T_n$  operate on  $\sum_n \oplus H_n$ . For arbitrary  $m$  it is clear that

$$\eta(\sigma(T_m)) \cap \eta\left(\sigma\left(\sum_{n \neq m} \oplus T_n\right)\right) = \emptyset.$$

Also, because of the arrangement of the spectra of the  $T_n$ 's it can be shown that for each  $m$  we have

$$\eta(\sigma(T)) = \eta(\sigma(T_m)) \cup \eta\left(\sigma\left(\sum_{n \neq m} \oplus T_n\right)\right).$$

Repeated use of Theorem 1 then gives the result that if  $M$  is an invariant subspace of  $T$ , then  $M = \sum_n \oplus M_n$ , where  $M_n$  is invariant under  $T_n$ . Hence the lattice of invariant subspaces for  $T$  is isomorphic to  $\mathbf{X}_n L_n$ .

Theorem 2 allows us to construct new examples of S-lattices. For example, if we let  $L_n$  be a lattice known to be attainable on a finite-dimensional space, for  $n = 1, 2, 3, \dots$ , then  $\mathbf{X}_n L_n$  is attainable on an infinite-dimensional space. Also the fact that the closed unit interval is an S-lattice, (attained by the Volterra integral operator—see [2] or [5]), implies that the closed unit square (with the product order) is an S-lattice.

An example of particular interest is  $(\omega+1) \times (\omega+1)$ . Donoghue [2] has shown that  $\omega+1$  is attainable by a compact operator, and thus our construction in Theorem 2 furnishes a polynomially compact operator with lattice  $(\omega+1) \times (\omega+1)$ . The second author has shown, however, that  $(\omega+1) \times (\omega+1)$  is not attainable by a compact operator.

#### REFERENCES

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