MODULAR REPRESENTATION ALGEBRAS

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Let $G$ be a cyclic $p$-group, $K$ a field of characteristic $p$, and $KG$ the group algebra of $G$ over $K$. The representation ring $a(KG)$ is generated by symbols $[M]$, one for each isomorphism class $\{M\}$ of finitely generated left $KG$-modules, with relations

$$[M] + [M'] = [M \oplus M'], \quad [M][N] = [M \otimes_K N].$$

The representation algebra $A(KG)$ is defined as $C \otimes_{\mathbb{Z}} a(KG)$, where $\mathbb{Z}$ is the ring of rational integers, $C$ the complex field. The aim of this note is to give a simple proof of the following theorem of Green [1].

**Theorem.** The representation algebra $A(KG)$ is semisimple.

Since $G$ is a cyclic $p$-group, the algebra $A(KG)$ is finite dimensional (and commutative), having $C$-basis $\{v_1, \ldots, v_q\}$, where $q = [G: 1]$, and where $v_r = [V_r]$. Here, $V_r$ denotes the unique indecomposable $KG$-module of dimension $r$. We set $A_0 = R \otimes_{\mathbb{Z}} a(KG)$, where $R$ is the real field. Then $A(KG) = C \otimes_K A_0$, and it suffices to prove that $A_0$ is semisimple, or equivalently, that $A_0$ has no nonzero elements of square zero.

By the *components* of a module we mean the indecomposable summands in a direct sum decomposition of the module.

**Lemma 1** (ROTH [4], RALLEY [3]). The number of components of $V_r \otimes V_s$ is precisely $\min(r, s)$.

**Proof.** Let $H_r$ be the $r \times r$ matrix with 1's above the main diagonal and zeros elsewhere, let $E_r$ be the $r \times r$ identity matrix, and let $\lambda$ be an indeterminate over $K$. Then the number of components of $V_r \otimes V_s$ is the same as the number of invariant factors of $(\lambda E_r + H_r)^*$ different from 1. This easily yields the desired result.

Let us write

$$v_r v_s = \sum_{t=1}^{q} a_{rs} v_t, \quad 1 \leq r, s \leq q.$$

Then the coefficients $\{a_{rs}\}$ are nonnegative integers, and Lemma 1 asserts that

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\[ \sum_{t=1}^{q} a_{rt} = \min (r, s), \quad 1 \leq r, s \leq q. \]

**Lemma 2.** The quadratic form
\[ \sum_{r,s=1}^{q} \min (r, s)X_rX_s \]

is positive definite.

**Proof.** One verifies that the given form coincides with
\[ (X_1 + \cdots + X_q)^2 + (X_1 + \cdots + X_q)^2 + \cdots + X_q. \]

We now show that if \( u \in A_0 \) satisfies \( u^2 = 0 \), then necessarily \( u = 0 \).

Write \( u = \sum_{r=1}^{q} \alpha_r v_r, \alpha_r \in R \).

Then
\[ \sum_{r,s} \alpha_r \alpha_s a_{rs} = 0, \quad 1 \leq r \leq q. \]

Summing on \( t \), we obtain
\[ \sum_{r,s} \min (r, s) \alpha_r \alpha_s = 0, \]

so by Lemma 2, \( \alpha_r = 0 \) for \( 1 \leq r \leq q \). This completes the proof.

The above technique has also been used by Hannula [2].

**References**