ON THE SUMMABILITY OF THE DIFFERENTIATED 
FOURIER SERIES

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Dedicated to Professor A. Zygmund on the occasion of his 65th birthday
Communicated by H. Helson, July 21, 1966

A classical theorem of Fatou [2, p. 99] asserts that if \( f \in L(0, 2\pi) \) 
and the symmetric derivative of \( f \) at \( x_0 \),

\[
f'_s(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}
\]

exists, then the differentiated Fourier series of \( f \) is Abel summable to \( f(x_0) \) at \( x_0 \), or equivalently, if \( u(r, x) = a_0/2 + \sum (a_k \cos kx + b_k \sin kx)r^k \) 
is the associated harmonic function, then

\[
\lim_{r \to 1^-} u_x(r, x_0) = f'_s(x_0).
\]

Let us suppose that \( \phi \) is a real nonnegative function on an interval 
to the right of the origin, that \( \phi(0) = 0 \), and that \( \phi(t) = O(t) \) as \( t \to 0 \).
We say that a set is \( \phi \)-dense at a point \( p \) if

\[
m(E^c \cap I)/\phi(m(I)) \to 0
\]
as \( m(I) \to 0 \), \( I \) an interval containing \( p \). If \( \phi \) is the identity function, 
this reduces to ordinary metric density. In the case \( \phi(t) = t^\alpha \), we will 
say that \( E \) is \( \alpha \)-dense at \( p \). Proceeding in a manner entirely analogous 
to the classical definition of approximate limit and derivative, we 
say that

\[
\phi-lim_{t \to t_0} g(t) = a
\]
if for every \( \epsilon > 0, E_\epsilon = \{ t \mid |g(t) - a| < \epsilon \} \) is \( \phi \)-dense at \( t_0 \), and we define the \( \phi \)-approximate symmetric derivative,

\[
\phi-f'_\alpha(x_0) = \phi-lim_{h \to 0} (f(x_0 + h) - f(x_0 - h))/2h.
\]

We restrict our attention here to the case of most immediate interest, 
\( \alpha \)-density, and prove the following

**Theorem.** Suppose \( f \) is in \( L(0, 2\pi) \), of period \( 2\pi \), essentially bounded 
in a neighborhood of \( x_0 \), and, for some \( \alpha \geq 2 \), \( y = \alpha-f'_\alpha(x_0) \). Then the

1 Supported by National Science Foundation Grant No. GP-3987.
differentiated Fourier series of \( f \) is Abel summable to \( y \) at \( x_0 \). The value 2 cannot be replaced by a smaller value nor can essentially bounded be replaced by integrable.

Ikegami [1] has shown that \( f'_i \) cannot be replaced by \( f'_{ap} \) in Fatou's theorem, even if \( f \) is bounded. He introduced
\[
\alpha f'_{ap}(x_0) = \alpha \lim_{h \to 0^+} \frac{f(x_0+h)-f(x_0)}{h}
\]
and attempted to show that, for bounded \( f \), Fatou's theorem holds with this derivative if \( \alpha > 4 \). His argument, however, contains an error, and when it is corrected yields this result only for \( \alpha > 5 \).

Turning to the proof of our result, we may suppose that \( x_0 = 0 \), \( f(0) = 0 \), and also \( \alpha f''_{ap}(0) = 0 \) as in the classical case [2, p. 100–101]. For the Poisson kernel,
\[
P(r, t) = \frac{1}{2} \frac{1 - r^2}{1 - 2r \cos t + r^2},
\]
we have the estimates
\[
P(r, t) < C\eta/(\eta^2 + t^2), \quad |P_t(r, t)| < C\eta t/(\eta^4 + t^4),
\]
where \( \eta = 1 - r \) and, throughout this paper, \( C \) will denote a positive constant not necessarily the same at each occurrence. The first estimate here is well known; the other may be obtained in a similar manner.

We may assume \( \alpha = 2 \), for if \( \alpha f''_{ap}(0) \) exists for some \( \alpha > 2 \), it also exists and has the same value for \( \alpha = 2 \).

There is a \( \delta_0 > 0 \) and an \( M > 0 \) such that \( |f(x)| \leq M \) a.e. in \( (-\delta_0, \delta_0) \). Now
\[
u_x(r, 0) = -\frac{1}{\pi} \int_0^\pi (f(t) - f(-t)) P_t(r, t) dt
\]
and, for any \( \delta \in (0, \delta_0) \), we may partition the interval of integration into \( (0, \delta) \), \( (\delta, \delta_0) \), and \( (\delta_0, \pi) \), denoting the absolute values of the above integral over these intervals by \( \delta_1 \), \( \delta_2 \), and \( \delta_3 \) respectively. We show that these values can be made arbitrarily small by choosing \( r \) sufficiently close to 1.

Clearly
\[
\delta_2 \leq 2M \int_\delta^\pi |P_t(r, t)| dt < C\eta \delta^{-2}
\]
and
Given an $\varepsilon > 0$, we set

$$E = \{ t \mid \left| f(t) - f(-t) \right| / 2t \geq \varepsilon \}.$$ 

Then

$$\mathcal{G}_1 \leq \left| \int_{E \cap (0, \delta)} \cdots \right| + \left| \int_{E \cap (0, \delta)} \cdots \right| = \mathcal{G}_1' + \mathcal{G}_1''$$

and we have

$$\mathcal{G}_1'' \leq \varepsilon \int_0^{\delta} 2t \left| P_t(r, t) \right| dt < -2\varepsilon \int_0^{\delta} tP_t(r, t) dt < C\varepsilon$$

by an integration by parts.

The estimation of $\mathcal{G}_1'$ is somewhat more difficult.

We now choose $\delta$ such that, for $t \in (0, \delta)$,

$$m(E \cap (0, t)) < \varepsilon t^2.$$ 

Let $t_1 = \delta$ and choose $t_k$, $k = 2, 3, \ldots$, in $(0, \delta)$, decreasing and converging to zero. We let $I_k = (t_{k+1}, t_k)$. Then

$$\mathcal{G}_1' \leq MC\eta \int_{E \cap (0, \delta)} t/(\eta^4 + t^4) dt$$

$$< C\eta \sum m(E \cap I_k)t_k/(\eta^4 + t_{k+1}^4) < C\eta \varepsilon \sum t_k/(\eta^4 + t_{k+1}^4).$$

Now let $t_k = \delta/2^{k-1}$. It is easily verified that

$$2^k \int_{I_k} t^2/(\eta^4 + t^4) dt > t_k^3/(\eta^4 + t_{k+1}^4)$$

for every $k$ and, therefore,

$$\mathcal{G}_1' < C\eta \varepsilon \int_0^{\delta} t^2/(\eta^4 + t^4) dt < C\varepsilon.$$ 

Thus

$$\left| u_{\varepsilon}(r, 0) \right| < C(\varepsilon + \eta + \eta^\delta - 2) < C\varepsilon.$$
if \( \eta \) is sufficiently small, the constant being independent of the choice of \( \varepsilon \).

Suppose now that \( \alpha \in [1, 2) \) and choose \( \beta \in (\alpha, 2) \). Let \( I_n = (1/2^n, 1/2^n + 1/2^{n+1}) \) and \( E = \bigcup I_n \). Then if \( 1/2^n < \ell \leq 1/2^{n-1} \), there exist positive constants \( C \) and \( C' \) such that

\[
C/2^{\beta n} < m(E \cap (0, \ell)) < C'/2^{\beta n}
\]

for every \( n \). Thus \( m(E \cap (0, \ell)) = o(t^\alpha) \) as \( t \to 0 \). If \( f = \chi_E \), the characteristic function of \( E \), then for sufficiently small \( \varepsilon > 0 \),

\[
\{ t \mid |f(t) - f(-\ell)|/2t | \geq \varepsilon \} = E
\]

and so

\[
\alpha = f'_{\text{ess}}(0) = 0.
\]

For \( 0 < a < b < \pi/2 \), it may be shown that

\[
- \int_a^b P_t(r, t) dt > C \frac{(a + b)(b - a)}{\eta^4 + b^4}.
\]

Thus, if \( \eta = 2^{-k} \), we have

\[
\varphi_k(r, 0) = - \frac{1}{\pi} \sum \int_{I_n} P_t(r, t) dt > - \frac{1}{\pi} \int_{I_{k+1}} P_t(r, t) dt
\]

\[
> C2^{-(\beta+2)k}/(2^{-4k} + 2^{-(k+1)} + 2^{-\beta(k+1)})^4
\]

\[
> C2^{(2-\beta)k} \to \infty
\]

as \( k \to \infty \), which shows that values of \( \alpha < 2 \) are inadmissible.

Finally suppose \( \alpha \geq 2, \beta > \alpha \), and define \( E \) as above. Now let \( f = \sum 2^{(\beta-1)n} \chi_{I_n} \). Then \( f \in L(0, 2\pi) \) and \( \alpha f'_{\text{ess}}(0) = 0 \). However,

\[
\varphi_k(r, 0) > - \int_{I_{k+1}} 2^{(\beta-1)(k+1)} P_t(r, t) dt
\]

\[
> C2^{(\beta-1)(k+1)} \cdot 2^{(2-\beta)k} = C2^k \to \infty
\]

as \( k \to \infty \), which shows that the requirement of essential boundedness cannot be removed.

REFERENCES


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