THE RADIAL HEAT EQUATION WITH POLE TYPE DATA

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1. Introduction. Recently, detailed studies have been undertaken relating to the solutions and expansions of solutions of the initial value problem

\[(a)\quad U_t(r, t) = \Delta_\mu U(r, t), \quad r > 0, \quad t > 0,\]

\[(b)\quad U(r, 0) = \phi(r)\]

with \(\Delta_\mu = D_r^2 + [(\mu - 1)/r]D_r\). Results have been obtained when \(\phi(r)\) is entire of growth \((1, \sigma)\) in \(r^2\) \([1], [3], [4]\) and these have been extended to the \(L_2\) theory in \([3]\). In this note, we state some results on the structures of solutions of (1) when the data function \(\phi(r)\) has a pole at \(r = 0\) but is otherwise entire. These structures are defined in terms of convolution integrals and the proofs are based on the Laplace transform formulation \([2]\) of solutions of (1) and the expansion theory referred to above. The details of the proofs will appear in a forthcoming paper that will also discuss logarithmic singularities.

We denote by \(U^\mu(r, t; \phi(r))\) the solution of (1) defined by

\[\int_0^\infty K_\mu(r, \xi; t) \phi(\xi) d\xi\]

with

\[K_\mu(r, \xi; t) = \frac{1}{2t} r^{1-\mu/2} \xi^{\mu/2} \exp \left[-(r^2 + \xi^2)/4t\right] I_{\mu/2-1}(r\xi/2t)\]

(See \([1], [4]\).) The abbreviation \(a = r^2/16t^2\) will be used in the statement of results.

2. Main results. Our first theorem relates to functions \(\phi(r)\) that are odd while the remaining results relate strictly to functions with poles.

**Theorem 1.** Let \(\phi(r) = r\psi(r)\) in which \(\psi(r)\) is an entire function of \(r^2\) of growth \((1, \sigma)\). For \(0 \leq t < 1/4\sigma\) and \(\mu > 2\),

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\[ U^\mu(r, t; \phi(r)) \]

\[ = \frac{r^{2-\mu} \exp \left[ -\frac{r^2}{4t} \right]}{\pi^{1/2}(4t)^{1/2-\mu}} \int_0^a (a - \xi)^{-1/2} \xi^{(\mu-2)/2} e^{4t} U^{\mu-1}(4t, t; \psi) d\xi. \]

**Theorem 2.** Let \( \phi(r) = r^{2-\mu-2\alpha} \psi(r) \) with \( 0 < \alpha < 1/2 \) and \( \psi(r) \) an entire function of \( r^2 \) of growth \( (1, \sigma) \). For \( 0 \leq t < 1/4\sigma \) and \( \mu > 2 \)

\[ U^\mu(r, t; \phi(r)) = \frac{r^{2-\mu} \exp \left[ -\frac{r^2}{4t} \right](4t)^{1/2-\alpha-1}}{\Gamma(\mu/2 + \alpha - 1)} \]

\[ \cdot \int_0^a \xi^{-\alpha}(a - \xi)^{\mu/2+\alpha-2} e^{4t} U^{\mu-2\alpha}(4t, t; \psi) d\xi. \]

Observe that the choice \( \alpha = 0 \) in Theorem 2 corresponds to the case in which the multiplier of \( \psi(r) \) is precisely the potential function for the Laplacian operator \( \Delta_\mu \). This theorem shows that the pole can be more badly behaved than the potential function. In fact, the following theorem shows that the pole can be as badly behaved as \( r^{\mu+\varepsilon} \) for arbitrary \( \varepsilon > 0 \) and still give rise to a classical solution.

**Theorem 3.** Let \( \phi(r) = r^{2-\mu-2\alpha} \left\{ A + r^2 \psi(r) \right\} \) in which \( \alpha \) is close to but less than \( 1, \mu/2 + \alpha > 2, A \) is a constant, and \( \psi(r) \) is an entire function of \( r^2 \) of growth \( (1, \sigma) \). For \( 0 \leq t < 1/4\sigma \),

\[ U^\mu(r, t; \phi) = \frac{r^{2-\mu} \exp \left[ -\frac{r^2}{4t} \right]}{(4t)^{1/2-\mu}} \]

\[ \cdot \left\{ \frac{A(4t)^{1-\alpha}}{\Gamma(\mu/2 + \alpha - 1)} \int_0^a \xi^{-\alpha}(a - \xi)^{\mu/2+\alpha-2} e^{4t} d\xi + \frac{(4t)^{2-\alpha}}{\Gamma(\mu/2 + \alpha - 2)} \right\} + \int_0^a \xi^{1-\alpha}(a - \xi)^{\mu/2+\alpha-3} e^{4t} U^{4-2\alpha}(4t, \sqrt{\xi}, t; \psi) d\xi \}

It follows, from the change of valuables \( \xi = a\sigma \), that

\[ \lim_{r \to 0; t \to 0} U^\mu(r, t; \phi) \]

exists in all of the above theorems. This simply states that the pole in the data function dissipates from the solution function.

Finally, as a corollary to Theorem 2 where \( \alpha = 0 \), we have the special result:

**Corollary 2.1.** Let \( \mu = 2m \) be an even integer with \( m \geq 2 \). Then
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\[
U^{2m}(r, t; r^2-2m+2j) = r^{2-2m}R_j^r(r, t) + (-1)^{i+1} \exp \left[-\frac{r^2}{4t}\right]
\]

(5)

\[
\sum_{k=0}^{m-2-j} \frac{(m-2-k)!}{k!(m-2-j-k)!} r^{2(1-m+k)}(4t)^{-k}, \quad 0 \leq j \leq m - 2.
\]

In this, \(R^{d-n}_\mu(r, t) = j!(4t)^{j}L_j^{(1-n/2)}(-r^2/4t)\) with \(L_j(x)\) the generalized Laguerre polynomial of degree \(j\) and index \(\nu\). In the case that \(\mu\) is even with \(\mu \geq 4\), we can divide the data into the pole type terms (finite in number) and the entire part. The corollary applies to the pole terms while the expansion theory in [1], [4] applies to the entire part.

REFERENCES


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