ON MANIFOLDS WITH INVOLUTION

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We consider a smooth involution \( \omega \) on a smooth closed \( n \)-manifold \( M \) from the bordism point of view, as in [2, Chapter IV]. We know that the fixed-point set of \( \omega \) is the disjoint union of submanifolds; let \( k \) be the maximum dimension of these. It is clear that if \( \omega \) is free, \( M \) bounds a 1-disk bundle over the orbit space. Now fix \( k \), and let \( n \), \( M \), and \( \omega \) vary. Conner and Floyd prove [2, Theorem (27.1)] that if \( M \) does not bound, \( n \) cannot be arbitrarily large. Their proof is nonconstructive, and fails to give an upper bound for \( n \). We obtain the precise bound.

**Theorem 1.** Suppose \( k \) is the maximum dimension of the fixed-point submanifolds of the smooth involution \( \omega \) on the closed nonbounding \( n \)-manifold \( M \). Then \( n \leq 5k/2 \) (if \( k \) is even) or \( n \leq (5k-1)/2 \) (if \( k \) is odd). Further, if we are given that the unoriented cobordism class of \( M \) is indecomposable, then \( n \geq 2k+1 \).

Examples in the extremal dimensions are easily constructed. Take homogeneous coordinates \((x_0, x_1, x_2, \ldots, x_i, x'_i, x'_1, \ldots, x'_i)\) on real projective \( 2i \)-space \( P_{2i} \) \((i>0)\), and define the involution \( \omega_i \) by

\[
\omega_i(x_0, x_1, \ldots, x_i, x'_i, \ldots, x'_i) = (x_0, x_1, \ldots, x_i, -x'_i, \ldots, -x'_i).
\]

Then the product involution \( \omega_i \times \omega_j \) on \( P_{2i} \times P_{2j} \) maps the hypersurface \( H_{2i,2j} \) defined by the equation

\[
x_0y_0 + x_1y_1 + \cdots + x_iy_i + x'_i y'_i + \cdots + x'_i y'_i = 0
\]

into itself, where for clarity we take coordinates \((y_0, y_1, \ldots, y_i, y'_i, \ldots, y'_i)\) on \( P_{2j} \), and assume \( i \leq j \). The fixed-point dimension of \( \omega_i \times \omega_j | H_{2i,2j} \) is found to be \( i+j-1 \). The manifold \( H_{2i,2j} \) has dimension \( 2i+2j-1 \) and its cobordism class \([H_{2i,2j}]_2\) is indecomposable if and only if the binomial coefficient

\[
\binom{i+j}{i}
\]

is odd. We can always choose \( i \) and \( j \) satisfying this condition and \( i+j=m \) whenever \( m \) is not a power of 2. As an example for the first assertion of Theorem 1, we take the product of many copies of the \( 5 \)-dimensional example \((H_{2,4}, \omega_1 \times \omega_2 | H_{2,4})\), with possibly one copy of \((P_2, \omega_i)\).
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We deduce Theorem 1 from Theorems 2 and 3 below, which are purely algebraic. They concern the bordism $J$-homomorphism

$$J_m: \mathcal{N}_i(BO(m)) \rightarrow \mathcal{N}_{i+m-1}(BO(1))$$

defined in [2, §25]. (It has only a tenuous connection with the Hopf-Whitehead $J$-homomorphism, which could be written

$$J: \pi_i(BO(m)) \rightarrow \pi_{i+m-1}(P_m).$$

Let us recall from [2] the bordism classification of manifolds with involution $(M, \omega)$. The normal bundle in $M$ of the $i$-dimensional fixed-point set of $\omega$ determines an element $\nu_i \in \mathcal{N}_i(BO(n-i))$. The main structure theorem (28.1) asserts that these elements characterize the bordism class of $(M, \omega)$, and are arbitrary, subject only to the condition

$$\sum_i J_{n-i} \nu_i = 0.$$

We stabilize $J_n$ by defining a homomorphism

$$J: \mathcal{N}_*(BO) \rightarrow \mathbb{R}[[\theta]],$$

where $\mathbb{R}[[\theta]]$ is the ring of homogeneous formal power series over $\mathbb{R}$ in an indeterminate $\theta$ of degree $-1$. Given $\alpha \in \mathcal{N}_i(BO)$, we put $J\alpha = \sum \alpha \theta^r$. To define the coefficient $\alpha_r \in \mathbb{R}$, we first lift $\alpha$ to $\alpha' \in \mathcal{N}_i(BO(m))$ for some $m \geq r-i+1$, and put $\alpha_r = \varepsilon \Delta^{i+m-r-1} J_m \alpha'$, where $\Delta$ is the bordism Smith homomorphism defined in [2, §26], and $\varepsilon: \mathcal{N}_*(BO(1)) \rightarrow \mathbb{R}$ is the canonical augmentation; $\alpha_r$ is independent of $m$ by Theorem (26.4) of [2]. It is more natural to define $J: F \rightarrow \mathbb{R}[[\theta]]$, where $F = \bigoplus \mathcal{N}_i(BO)$, by extending linearly. Then the elements $\nu_i$, when included in $\mathbb{R}_*(BO)$, may be added in $F$ to form an element $\nu \in F$. The relation $\sum_i J_{n-i} \nu_i = 0$, and also (24.2) of [2], are combined in the following formula.

THEOREM 2. $J\nu = [M]\theta^n + \text{terms with higher powers of } \theta$.

Now the cross product of vector bundles makes $F$ into an ungraded polynomial ring. It is easy to see that

$$J1 = 1 + [P_2]_2 \theta^2 + [P_4]_4 \theta^4 + [P_6]_6 \theta^6 + \cdots.$$

Therefore we define $J': F \rightarrow \mathbb{R}[[\theta]]$ by setting $J'\alpha = (J\alpha) \cdot (J1)^{-1}$, so that $J'1 = 1$. We may clearly replace $J$ by $J'$ in Theorem 2. We are interested in the case when

$$\nu \in F_k = \bigoplus_{i=0}^{\infty} \mathcal{N}_i(BO) \subseteq F.$$
THEOREM 3. $J': R 	o R[[\theta]]$ is a ring homomorphism. Further, we can find systems of polynomial generators $z_i \in R_i$ for $R$ and $x_i$ for $F$, for each $i$ not of the form $2^a - 1$, such that

(a) $J'x_i = z_i\theta^i + \text{terms with higher powers of } \theta$,

(b) If we assign to $x_i$ the weight $i/2$ ($i$ even) or $(i-1)/2$ ($i$ odd), then $F_k$ consists of all polynomials of weight $\leq k$ in the elements $x_i$.

There ought to be a direct geometric proof that $J'$ is a ring homomorphism.

The computation of $J_n$, and hence of $J'$, is in principle known from [2, Chapter IV]. All that is lacking is a certain amount of technique. Full details will appear in [1].

REFERENCES


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