HIGHER RANK CLASS GROUPS

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Let $A$ be a noetherian ring which is locally Macaulay. For each integer $i \geq 0$, groups $C_i(A)$ and $W_i(A)$ are defined, each sequence of groups generalizing to higher dimensions the usual class group of an integrally closed noetherian domain. $C_i(A)$ is called the $i$th class group of $A$, and $W_i(A)$ is called the $i$th homological class group of $A$. The main purpose of this note is to show that both sequences of groups have properties analogous to the class group of a Noetherian integrally closed integral domain, and finally to establish a connection between them.

1. Throughout this section $A$ is a commutative noetherian ring which is locally Macaulay. A set of elements $x_1, \ldots, x_s$ is an $A$-sequence of length $s$ if $x_1A + \cdots + x_sA \neq A$ and $x_1A + \cdots + x_iA: x_{i+1} = x_1A + \cdots + x_iA$ for $i = 0, 1, \ldots, s - 1$. Count the empty set as an $A$-sequence of length 0 and specify that it generate the zero ideal of $A$.

Note that if $x_1, \ldots, x_s$ is an $A$-sequence of length $s$, then $x_1A + \cdots + x_sA$ is an unmixed ideal of $A$ of height $s$.

For each $i \geq 0$, form the free abelian group on the generators $(p)$ where $p$ is a height $i$ prime ideal of $A$. This group will be denoted by $D_i(A)$. For each $A$-sequence $x_1, \ldots, x_i$, consider the element $\sum e(x_1, \ldots, x_i | A p)(p)$ of $D_i$ (here $e(y_1, \ldots, y_i | M)$ denotes the multiplicity of $y_1A + \cdots + y_iA$ on $M$). Let $R_i$ designate the subgroup of $D_i$ generated by all such elements. Set $C_i(A) = D_i(A)/R_i$ and call $C_i(A)$ the class group of rank $i$ for $A$. Denote the image of $(p)$ in $C_i(A)$ by $\text{cl}(p)$. Set $C.(A) = \oplus C_i(A)$.

EXAMPLES. $C_0(A)$ is always finitely generated. $C_0(A)$ is finite if and only if $(0)$ is a primary ideal of $A$. $C_0(A) = 0$ if and only if $A$ is a domain.

If $A$ is a Dedekind domain, then $C_1(A)$ is the ordinary ideal class group of $A$. More generally, if $A$ is integrally closed, then $C_1(A)$ is the class group of $A$ [1, §1, no. 10].

We have not been able to locate the following lemma in the literature.

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Lemma 1.1. Let $S$ be a multiplicatively closed subset of $A$. If $y_1, \ldots, y_i$ is an $A_S$-sequence, then there is an $A$-sequence $x_1, \ldots, x_i$ such that $\sum y_i A_S = \sum x_i A_S$.

Theorem 1.2. (Cf. [1, Proposition 17, §1, no. 10].) Let $S$ be a multiplicatively closed subset of $A$. Then for each $i \geq 0$, there is an epimorphism $C_i(A) \to C_i(A_S)$ deduced from $\langle p \rangle \to 0$ if $p \cap S \neq \emptyset$ and $\langle p \rangle \to \langle p A_S \rangle$ if $p \cap S = \emptyset$. The kernel is generated by $\{cl(p)\}$ where $ht(p) = i$ and $p \cap S \neq \emptyset$.

Corollary 1.3. (Cf. [4, Lemma 1.7].) If $p \cap S \neq \emptyset$ implies that $cl(p) = 0$, then the epimorphism of Theorem 1.2 is an isomorphism.

Corollary 1.4. If $C_i(A_S) = 0$, then $C_i(A)$ is generated by $\{cl(p)\}$ where $ht(p) = i$ and $p \cap S \neq \emptyset$.

Corollary 1.5. There is an epimorphism $C_i(A) \to \oplus_{ht(p) = i} C_i(A_p)$ deduced from $(p) \to (p A_p)$.

Theorem 1.6. If $x_1, \ldots, x_k$ is an $A$-sequence, then there is a homomorphism $C_i(A / \sum x_i A) \to C_{i+k}(A)$ whose image is the subgroup of $C_{i+k}(A)$ generated by $\{cl(p)\}$ where $ht(p) = i+k$ and $p \supseteq \sum x_i A$.

With Theorem 1.2, this yields

Corollary 1.7. Suppose that $x$ is an $A$-sequence. Then the sequence

$$C_i(A/xA) \to C_{i+1}(A) \to C_{i+1}(A[x^{-1}]) \to 0$$

is exact.

An application of the associative law for multiplicities yields

Theorem 1.8. If $ht(p) = k$ and $cl(p) = 0$, then there is a homomorphism $C_i(A/p) \to C_{i+k}(A)$ whose image is the subgroup of $C_{i+k}(A)$ generated by the $cl(q)$ where $ht(q) = i+k$ and $q \supseteq p$.

Using techniques similar to those of [2, Proof of Proposition 7-1] we get

Lemma 1.9. Suppose that $C_i(A_p) = 0$ for each prime ideal $p$ of height $i$ of $A$. Then $C_{i+1}(A[X])$ is generated by $\{cl(q A[X])\}$ where $q$ ranges over the prime ideals of $A$ of height $i+1$.

Theorem 1.10. If $C_i(A_p) = 0$ for all prime ideals $p$ of $A$ of height $i$, then there is an epimorphism $C_{i+1}(A) \to C_{i+1}(A[X])$.

Corollary 1.11. (Cf. [1, Corollary to Theorem 2].) $C_i(A) = 0$ implies $C_i(A[X]) = 0$. 

**Corollary 1.12.** If \( F \) is a field, then \( C_r(F[X_1, \ldots, X_n]) = 0 \).

**Corollary 1.13.** Let the Krull dimension of \( A \) be \( n < \infty \). Suppose that \( C_n(p) = 0 \) for each prime ideal \( p \) of \( A \) of height \( n \). Then \( C_{n+1}(A[X]) = 0 \).

A theorem similar to Theorem 1.10 is

**Theorem 1.14.** Let \( A \) and \( B \) be finitely generated over a field \( F \). Suppose, that for each \( i \geq 0 \), \( C_i(A) = 0 \) for any prime ideal \( p \) of height \( i \) of \( A \), and that \( C_i(K \otimes_F B) = 0 \) for any overfield \( K \) of \( F \). Then there is an epimorphism \( C_i(A) \to C_i(A \otimes_F B) \) given by \( \text{cl}(p) \to \text{cl}(p \otimes_F B) \). In particular, \( C_i(A) = 0 \) implies \( C_i(A \otimes_F B) = 0 \).

**Theorem 1.15.** For \( i \geq 1 \), \( C_i(A_1 \oplus A_2) = C_i(A_1) \oplus C_i(A_2) \).

2. Let \( A \) be a commutative noetherian ring. The hypotheses on \( A \) in §1 need not be assumed in order to define the groups \( W_i(A) \). The reader is referred to [3] for the \( K \)-theory needed here.

Let \( \mathfrak{M}_i(A) = \mathfrak{M}_i \) denote the category of finitely generated \( A \)-modules \( M \) such that \( M_p = 0 \) for all prime ideals \( p \) of \( A \) with \( ht(p) < i \).

Then \( \mathfrak{M}_i \) is a Serre subcategory of \( \mathfrak{M}_j \) for all \( j > i \). Let \( K^i(\mathfrak{C}) \) denote the \( i \)th Grothendieck group of the category \( \mathfrak{C} \) for \( i = 0, 1 \). If \( \mathfrak{C} \in \mathfrak{C} \), then \( \gamma(C) \) denotes the image (or class) of \( C \) in \( K^0(\mathfrak{C}) \).

**Proposition 2.1.** \( K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \) is isomorphic to \( D_i(A) \), the isomorphism being given by the length function.

Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \rightarrow & K^0(\mathfrak{M}_i) \\
\downarrow & & \downarrow \\
K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \rightarrow & K^0(\mathfrak{M}_{i+1}) & \rightarrow & K^0(\mathfrak{M}_i) & \rightarrow & D_i(A) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & = \\
K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \rightarrow & D_{i+1}(A) & \rightarrow & K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \rightarrow & D_i(A) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & = \\
0 & & 0 & & \end{array}
\]

Because each element in \( \mathfrak{M}_i/\mathfrak{M}_{i+1} \) has finite length, each of the rows is exact. The columns are also exact.

Since the group \( D_i(A) \) is free, the kernels of \( g \) and \( g' \) in the above diagram are direct summands of their respective domains. For each
\( i \geq 0 \) define the group \( Z_{i+1}(A) \), and the \textit{homological class group of rank} \( i+1 \), \( W_{i+1}(A) \), to be the kernels of \( g \) and \( g' \) respectively. Since the rows are exact this is the same as saying that \( Z_{i+1}(A) \) is the image of \( f \) and \( W_{i+1}(A) \) is the image of \( f' \). Moreover

\[
K^0(\mathfrak{M}_i) = Z_{i+1}(A) \oplus D_i(A)
\]

and

\[
K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) = W_{i+1}(A) \oplus D_i(A).
\]

The results of [3] yield

**Proposition 2.2.** \( K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \) is isomorphic to the direct sum of the groups of units of \( A_p/pA_p \), \( ht(p) = i \). Consequently the kernel of \( f' \) is generated by the \( \gamma(A/p+xA) \), \( x \in \mathfrak{p} \), as \( \mathfrak{p} \) ranges over the prime ideals of \( A \) of height \( i \), and hence \( W_{i+1}(A) \) is \( D_{i+1}(A) \) modulo the subgroup generated by these.

By convention \( W_0(A) = 0 \). Set \( W^*(A) = \bigoplus W_i(A) \).

Diagram chasing will give

**Theorem 2.3.** Let \( S \) be a multiplicatively closed subset of \( A \). For each \( i \) there is an epimorphism \( W_i(A) \rightarrow W_i(A_S) \) induced by the functor \( A_S \otimes A \rightarrow A_S \). The kernel is generated by \( \gamma(A/p) \) with \( p \subseteq S \neq \emptyset \), \( ht(p) = i \).

**Corollary 2.4.** If for each prime ideal \( \mathfrak{p} \) of \( A \) of height \( i \) with \( \mathfrak{p} \subseteq S \neq \emptyset \), \( \gamma(A/p) = 0 \) in \( K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \) then the epimorphism of Theorem 2.3 is an isomorphism.

**Corollary 2.5.** If \( W_i(A_S) = 0 \), then \( W_i(A) \) is generated by \( \{ \gamma(A/p) \} \), \( ht(p) = i \), \( \mathfrak{p} \subseteq S \neq \emptyset \).

**Corollary 2.6.** The functors \( A_p \otimes A \rightarrow A_p \) induce an epimorphism \( W_i(A) \rightarrow \bigoplus_{ht(p) = i} W_i(A_p) \).

**Theorem 2.7.** Let \( A \) be locally Macaulay, \( I \) an unmixed ideal of height \( k \). Then there is a homomorphism

\[
W_i(A/I) \rightarrow W_{i+k}(A)
\]

induced by considering each \( A/I \)-module as an \( A \)-module. The image is generated by the \( \gamma(A/p) \), \( \mathfrak{p} \) a prime ideal of height \( i+k \) containing \( I \).

Using Theorems 2.3 and 2.7 one gets

**Theorem 2.8.** Let \( x \) be an \( A \)-sequence, \( A \) a locally Macaulay ring.

Then the sequence
\[ W_i(A/xA) \to W_{i+1}(A) \to W_{i+1}(A[x^{-1}]) \to 0 \]

is exact.

**Theorem 2.8.** The functor \( A \otimes_A \) induces an epimorphism \( W_i(A) \to W_i(A[X]) \). Furthermore \( W_{n+1}(A[X]) = 0 \) if the Krull dimension of \( A \) is \( n < \infty \).

**Corollary 2.9.** \( W^\ast(A) = 0 \) implies \( W^\ast(A[X]) = 0 \).

**Corollary 2.10.** \( W^\ast(F[X_1, \ldots, X_n]) = 0 \) when \( F \) is a field.

**Theorem 2.11.** Let \( A_1 \) and \( A_2 \) be two rings. Then
\[ W_i(A_1 \oplus A_2) = W_i(A_1) + W_i(A_2). \]

3. It is natural to ask if \( C_i(A) = W_i(A) \) when both are defined. There are several results in this direction.

**Theorem 3.1.** \( W_i(A) \) is a homomorphic image of \( C_i(A) \).

**Theorem 3.2.** If \( C_i(A) = 0 \), then \( W_{i+1}(A) = C_{i+1}(A) \).

**Corollary 3.3.** If \( A \) is a domain, then \( W_1(A) = C_1(A) \).

**Corollary 3.4.** \( C^\ast(A) = 0 \) if, and only if, \( A \) is an integral domain and \( W^\ast(A) = 0 \).

For an example which shows that in general \( W^\ast(A) \neq C^\ast(A) \) let \( Q \) be a primary ring which is not a field and set \( A = Q[X] \). Then \( W_1(A) = 0 \) while \( C_1(A) \) is an infinite group.

**References**


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