EQUIVARIANT STABLE STEMS

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Let $S^n(r)$ denote the $n$-sphere with a linear involution having a fixed point set of codimension $r$, where $0 \leq r \leq n$. We pick some fixed point as a base point and consider the set $[S^n(r); S^k(l)]$ of base point preserving equivariant homotopy classes of maps from $S^n(r)$ to $S^k(l)$. This has a natural group structure for $n-r \geq 1$ and is abelian if $n-r \geq 2$.

There is a suspension functor $S$ without action and one $\Sigma$ with action (that is, the reduced join with $S^i(0)$ and $S^i(1)$ respectively). These induce homomorphisms

$$[S^{n+1}(r); S^{k+1}(l)] \xrightarrow{S} [S^n(r); S^k(l)] \xrightarrow{\Sigma} [S^{n+1}(r+1); S^{k+1}(l+1)].$$

It can be shown that $S$ is an epimorphism when $n \leq 2k-1$ and $n-r \leq 2(k-l)-1$ and is an isomorphism if the strict inequalities hold. Similarly, $\Sigma$ is an epimorphism when $n \leq 2k-1$ and $n-r \leq k-1$ and is an isomorphism if the strict inequalities hold. By passing to the $S$-limit we define

$$\pi_n(r; l) = \lim_{k} [S^{n+k}(r); S^k(l)].$$

$\Sigma$ induces $\Sigma: \pi_n(r; l) \rightarrow \pi_n(r+1; l+1)$ which is an epimorphism when $n \leq r-1$ and an isomorphism when $n \leq r-2$. By passing to the $\Sigma$-limit we define

$$\pi_{n,k} = \lim_{l} \pi_n(l+k; l).$$

There is the forgetful functor $\psi$ (forgetting equivariance) and the fixed point set functor $\phi$ which yield homomorphisms

$$\pi_n \xleftarrow{\psi} \pi_n(r; l) \xrightarrow{\phi} \pi_{n-r+l}$$

where $\pi_n$ denotes the classical $n$-stem. For the doubly stable groups these become

$$\pi_n \xleftarrow{\psi} \pi_{n,k} \xrightarrow{\phi} \pi_{n-k}.$$
It is important to consider the generalization of these groups defined by

\[ \pi_n(r, q; t) = \lim_{k} \left[ S^{n+k}(r)/S^{n+k-r}(q); S^{k}(t) \right] \]

and we single out the case \( q = 0 \) by the special notation

\[ \pi_n^*(r; t) = \pi_n(r, 0; t). \]

If \( r \geq q \geq p \) there is an exact sequence

\[ \cdots \rightarrow \pi_n(r, q; t) \rightarrow \pi_n(r, p; t) \rightarrow \pi_{n-1}(r, q; t) \rightarrow \cdots \]

and also a similar exact sequence with \( p \) deleted. A special case of interest is that for which \( q = r - 1 \):

\[ \cdots \rightarrow \pi_n(r, p; t) \rightarrow \pi_{n-1}(r-1, p; t) \rightarrow \cdots \]

(\( p \) may be deleted here). Also of interest is the case in which \( p \) is deleted and \( q = 0 \):

\[ \cdots \rightarrow \pi_n^*(r; t) \rightarrow \pi_{n-1}(r; t) \rightarrow \cdots \]

(The maps \( \psi \) in (5) and \( \phi \) in (6) are the forgetful and fixed point homomorphisms respectively.)

The \( \Sigma \)-suspension yields an isomorphism

\[ \Sigma: \pi_n(r, q; t) \rightarrow \pi_n(r + 1, q + 1; t + 1) \]

in all cases.

Let \( \Phi(n) \) be the number of integers \( k \) with \( 0 < k \leq n \) and \( k \equiv 0, 1, 2, 4 \) (modulo 8). Our main general result is:

**Theorem.** If \( j \) is divisible by \( 2^\Phi(r-1) \) then there is an isomorphism

\[ \lambda_j: \pi_n(r, q; t) \rightarrow \pi_n(r, q; t + j). \]

If \( 2^\Phi(r-1) \mid j \) then \( \lambda_j \) commutes with \( \psi \). In particular,

\[ \pi_n^*(r; t) \approx \pi_n^*(r; t + j) \quad \text{for } 2^\Phi(r-1) \mid j \]

and

\[ \pi_{n,k}^* \approx \pi_{n,k-i}^* \quad \text{for } 2^\Phi(n+1) \mid j. \]

Moreover, the period \( 2^\Phi(n+1) \) for \( \pi_{n,k}^* \) is best possible when \( n + 1 \equiv 0, 1, 2, 4 \) (modulo 8).
Using this result, (7), Spanier-Whitehead duality and results of Atiyah [1] and James [2] it is easy to show that there is an isomorphism

\[ \pi^n_{k,r} \cong \pi^n_{n}(r; r + k) \]  

when \( n < k - 1 \).

Here \( \pi^n_{k,r} = \pi_{k+n}(V_{k+r,r}) \) where \( V_{k+r,r} \) is the Stiefel manifold \( O(k+r)/O(k) \). In particular, if \( n < k - 1 \) and \( n < r - 1 \), \( \pi^n_{k,r} \cong \pi^n_{n-2} \).

The periodicity, in \( k \), of the \( \pi^n_{k,r} \) which results from (8) is known and is due to James [2]. Also see [3].

The groups \( \pi^n_{k,r} \) have been calculated by Paechter [4] for \( n \leq 5 \). Our methods, which are aided by the relationships between \( \pi^n_{n}(r; r + k) \) and \( \pi^n_{n}(r; r + k) \), allow us to calculate the groups \( \pi^n_{n}(r; r + k) \) for \( n \leq 8 \) (with a few ambiguities) and their orders for \( n \leq 10 \).

Also the homomorphisms \( \varphi \) and \( \Phi \) are determined in roughly this range. We shall comment here only on \( \Phi \). First, there is the general result:

**Proposition.** \( \Phi: \pi_{n,k} \rightarrow \pi_{n-k} \) is onto if \( n \geq 2k \). It is also onto, and splits, if \( k \leq 0 \). If \( n < 0 \), \( \Phi \) is an isomorphism.

The calculations yield the following special results:

\[ \Phi: \pi_{n,n} \rightarrow 2^n \pi_0 \]  

for \( 1 \leq n \leq 4 \),

\[ \Phi: \pi_{5,5} \rightarrow 16 \pi_0 \],

\[ \text{Im}\{ \Phi: \pi_{n,n} \rightarrow \pi_0 \} \subset 16 \pi_0 \]  

for \( n \geq 4 \).

There are also exact sequences:

\[ 0 \rightarrow \mathbb{Z}_2 \rightarrow \pi_{6,6} \rightarrow 16 \pi_0 \rightarrow 0, \]

\[ 0 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \pi_{7,7} \rightarrow 16 \pi_0 \rightarrow 0. \]

\[ 0 \rightarrow \mathbb{Z}_8 \rightarrow \pi_{8,8} \rightarrow 16 \pi_0 \rightarrow \mathbb{Z}_2. \]

For \( n - k = 1, 2, 3 \) we obtain

\[ \Phi: \pi_{n,n-1} \rightarrow \pi_1 \]  

is \{onto for \( n \leq 3 \), zero for \( n \geq 4 \);\}

\[ \Phi: \pi_{n,n-2} \rightarrow \pi_2 \]  

is \{onto for \( n \leq 6 \), zero for \( n \geq 7 \);\}
\[ \phi: \pi_{n,n-3} \to \pi_3 \begin{cases} \text{onto } \pi_3 \text{ for } n \leq 7, \\ \text{onto } 2\pi_3 \text{ for } n = 8, \\ \text{onto } 4\pi_3 \text{ for } n = 9, \\ \text{onto } 8\pi_3 \text{ for } n \geq 10. \end{cases} \]

The kernel of \( \phi \) can be calculated in the range \( n \leq 8 \). (As far as we are aware the only previously known result in this direction was that \( \phi: \pi_{n,n} \to \pi_0 \) has image contained in \( 2\pi_0 \) for \( n \geq 1 \); known since it is an easy consequence of Smith theory.)

The calculations of the \( \pi_{n,k} \) are accomplished mainly through the use of spectral sequences associated with the exact sequence (4).

The only difficulties in calculating the \( \pi_{n,k} \) are in dealing with the \( 2\)-primary components. In fact, if \( \mathcal{C} \) denotes the class of finite \( 2\)-groups we can prove that

\[ \phi: \pi_{n,k} \overset{\approx}{\to} \pi_{n-k} \mod \mathcal{C} \text{ if } k \text{ is odd}, \]
\[ \phi \oplus \psi: \pi_{n,k} \overset{\approx}{\to} \pi_{n-k} \oplus \pi_n \mod \mathcal{C} \text{ if } k \text{ is even.} \]

An interesting corollary of the proof of the periodicity theorem is worth mentioning here. Let \( T_k \) be the matrix

\[ \begin{pmatrix} -I_k & 0 \\ 0 & I \end{pmatrix} \in O = \bigcup_{m=1}^{\infty} O(m). \]

Consider the antipodal involution \( A \) on \( S^n \) and the left translation by \( T_k \) on \( O \). Then we can show that an equivariant map

\[ (S^n, A) \to (O, T_k) \]

exists if and only if \( 2^{s(n)} | k \). The "if" part is easy and, in fact, when \( 2^{s(n)} | k \) we construct an equivariant map

\[ (S^n, A) \to (O(k), -I). \]

The proofs of these results will be published elsewhere.

*Note added in proof.* It has been brought to our attention that the groups \( \pi_n^2 \) have been calculated for \( n \leq 13 \) and \( r \) large by C. S. Hoo and M. E. Mahowald. Their results are tabulated in Bull. Amer. Math Soc. 71 (1965), 661–667. Unfortunately, it does not appear that their methods could give any information on the fixed point homomorphism. However, a comparison of their results with our methods does strongly indicate the conjecture that the image of \( \phi: \pi_{n,n} \to \pi_0 \) is \( a_n \pi_0 \) where \( a_8 = a_9 = 2^5 \), \( a_{10} = 2^6 \), \( a_{11} = 2^7 \), \( a_{12} = a_{13} = a_{14} = 2^8 \). (For \( n \leq 7 \), \( a_n \) is given above.)
REFERENCES

3. M. Mahowald, A short proof of the James periodicity of $\pi_{k+p}(V_{k+m,m})$, Proc. Amer. Math. Soc. 16 (1965), 512.

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