A GEOMETRIC PROOF OF RYLL-NARDZEWSKI'S
FIXED POINT THEOREM

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In [4], Ryll-Nardzewski gave what he called an ‘old-fashioned’ proof of his famous fixed point theorem. The purpose of the present note is to give an even more old-fashioned proof of the fixed point theorem. In fact, our proof uses nothing more than a category argument and the classical Krein-Milman theorem. Our terminology and notation shall be those of Kelley, Namioka et al. [2]. The following geometric lemma is essential to our proof of Ryll-Nardzewski’s fixed point theorem. In case the space $E$ and the pseudo-norm $p$ in the lemma are a Banach space and its norm respectively, the lemma is an easy consequence of Lindenstrauss’s work [3].

**Lemma.** Let $(E, 3)$ be a locally convex Hausdorff linear topological space, let $K$ be a nonempty 3-separable, weakly compact, convex subset of $E$, and let $p$ be a continuous pseudo-norm on $E$. Then for each $\epsilon > 0$, there is a closed convex subset $C$ of $K$ such that $C \subseteq K$ and $p\text{-diam}(K - C) \leq \epsilon$, where, for any subset $X$ of $E$, $p\text{-diam}(X) = \sup\{p(x - y) : x, y \in X\}$.

**Proof.** Let $S = \{x : p(x) \leq \epsilon/4\}$; then $S$ is a weakly closed convex body. Let $D$ be the weak closure of the set of all extreme points of $K$. Since $K$ is 3-separable, a countable number of translates of $S$ cover $K$ and hence $D$. Since $D$ is weakly compact, it is of the second category in itself with respect to the relative weak topology. Therefore there are a point $k$ of $K$ and a weakly open subset $W$ of $E$ such that $(S + k) \cap D \supset W \neq \emptyset$. Let $K_1$ be the closed convex hull of $D \cap W$, and let $K_2$ be the closed convex hull of $D \cap W$. Then, by the Krein-Milman theorem and the compactness of $K_1$ and $K_2$, $K$ is the convex hull of $K_1 \cup K_2$. Furthermore $K_1 \neq K$. For, otherwise, by Theorem 15.2 of [2], $D \cap W$ would contain all the extreme points of $K$, contradicting the fact that $W \cap D \neq \emptyset$. Obviously $p\text{-diam}(K_2) \leq \epsilon/2$. Now let $r$ be a real number in $(0, 1]$ and let $f_r$ be the map $K_1 \times K_2 \times [r, 1] \to K$ defined by $f_r(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$. Then clearly the image $C_r$ of $f_r$ is weakly closed, and it is easy to check that

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1 After the draft of the present note was completed we learned that Professor J. L. Kelley knew independently that a lemma of this sort was needed for a proof of Ryll-Nardzewski’s fixed point theorem. Thus he was able to give a short proof of the fixed point theorem for Banach spaces using Lindenstrauss’s result.
$C_r$ is convex. Moreover $C_r \neq K$. For, if $C_r = K$, then each extreme point $z$ of $K$ is of the form $z = \lambda x_1 + (1 - \lambda)x_2$, $x_i \in K_i$, $\lambda \in [r, 1]$. This would imply that each extreme point of $K$ is in $K_1$ or $K = K_1$, contradicting $K_1 \neq K$. Finally, if $y \in K \sim C_r$, then $y$ is of the form $y = \lambda x_1 + (1 - \lambda)x_2$, $x_i \in K_i$, $\lambda \in [0, r)$. It follows that $p(y - x_2) = \lambda p(x_1 - x_2) \leq rd$, where $d = p_{\text{diam}}(K) < \infty$. Since $p_{\text{diam}}(K_2) \leq e/2$, we have $p_{\text{diam}}(K \sim C_r) \leq e/2 + 2rd$. Therefore if we let $C = C_r$ for $r = e/4d$, the proof of the lemma is complete.

Let $Q$ be a subset of a locally convex space $E$ and let $S$ be a semigroup of transformations of $Q$ into $Q$. The semigroup $S$ is called noncontracting if $0$ does not belong to the closure of $\{ T x - T y : T \in S \}$ whenever $x \neq y$ and $x, y \in Q$. Clearly $S$ is noncontracting if and only if, for $x, y \in Q$ with $x \neq y$, there is a continuous pseudo-norm $p$ (depending on $x$ and $y$) on $E$ such that $\inf \{ p(T x - T y) : T \in S \} > 0$.

**Theorem (Ryll-Nardzewski).** Let $Q$ be a nonempty, weakly compact, convex subset of a locally convex Hausdorff linear topological space $E$, and let $S$ be a noncontracting semigroup of weakly continuous affine maps of $Q$ into itself. Then there is a common fixed point of $S$ in $Q$.

(The following proof is not the most direct one. However it establishes an additional interesting fact concerning fixed points, also due to Ryll-Nardzewski [4]: When $S$ is finitely generated, the problem of finding a common fixed point of $S$ can be reduced to that of a single operator.)

**Proof.** By a familiar compactness argument, it is sufficient to prove that each finite subset of $S$ has a common fixed point in $Q$. Therefore we may assume that $S$ is generated by $T_1, T_2, \ldots, T_r$.

Let $T_0 = (T_1 + T_2 + \cdots + T_r)/r$. Then $T_0$ is a weakly continuous affine map of $Q$ into itself; hence there is a fixed point $x_0$ of $T_0$ in $Q$ (see, for example, Théorème 1, Appendice of [1]). We will show that $T_ix_0 = x_0$ for $i = 1, 2, \ldots, r$. Assume that this is not the case. Then by throwing out those $T_i$'s for which $T_ix_0 = x_0$, we may assume that $T_ix_0 \neq x_0$ for $i = 1, 2, \ldots, r$. Since $S$ is noncontracting there is a continuous pseudo-norm $p$ on $E$ and $\varepsilon > 0$ such that

\[ p(TT_ix_0 - Tx_0) > \varepsilon \quad \text{for all } T \in S \text{ and } i = 1, \ldots, r. \]

Let $K$ be the closed convex hull of $\{ T x_0 : T \in S \}$. Then $K$ is a weakly compact, convex, separable subset of $E$. Hence, by the lemma, there is a closed convex subset $C$ of $K$ such that $C \neq K$ and

\[ C \neq K. \]
\( p \)-diam\((K \sim C) \leq \varepsilon \). Since \( C \neq K \), there is an element \( S \) in \( S \) such that \( Sx_0 \in K \sim C \). From \( T_0x_0 = x_0 \), we see that

\[
Sx_0 = (ST_1x_0 + ST_2x_0 + \cdots + ST_rx_0)/r.
\]

Hence \( ST_ix_0 \in K \sim C \) for at least one \( i \), since otherwise \( Sx_0 \in C \). It follows that \( p(ST_ix_0 - Sx_0) \leq p\)-diam\((K \sim C) \leq \varepsilon \), contradicting inequality (*). The proof of the theorem is therefore complete.

**Remark.** In the proof above \( T_0 \) could have been any convex combination \( \sum_{i=1}^{r} \lambda_i T_i \) with \( \lambda_i > 0 \).

**References**


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