A function $f(t)$ is called quasi-periodic if it can be represented in the form

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_m t)$$

where $F(\theta_1, \cdots, \theta_m)$ is a continuous function of period $2\pi$ in $\theta_v$, $v = 1, \cdots, m$. The numbers $\omega_1, \cdots, \omega_m$ are called the basic frequencies of $f(t)$. We shall denote by $A(\omega_1, \cdots, \omega_m)$ the class of all functions $f$ for which $F$ is real analytic. For simplicity of notation we set $\theta = (\theta_1, \cdots, \theta_m)$ and $\omega = (\omega_1, \cdots, \omega_m)$ (then $A(\omega_1, \cdots, \omega_m) = A(\omega)$ and $F(\theta_1, \cdots, \theta_m) = F(\theta)$).

The purpose of this note is to study the family of complex systems of differential equations:

$$z = \lambda z + \epsilon f(t, z, \bar{z}),$$

parametrized by $\lambda$, $f$ analytic in $z$, $\bar{z}$, and $f \in A(\omega)$—i.e. $f(t, z, \bar{z}) = g(\theta, z, \bar{z})$ where $g$ is $2\pi$-periodic in $\theta$—to determine the complex numbers, $\lambda$, for which there exists a solution $z = \phi(t, \epsilon) \in A(\omega)$.

For $\Re \lambda = 0$ there may be no solutions even in the linear case

$$\dot{z} = \lambda z + \epsilon g(\theta),$$

$$\dot{\theta} = \omega$$

because of resonance. It is well known that if $\Re \lambda \neq 0$ and $\epsilon > 0$ is small compared with $|\Re \lambda|$ then (1) always has a solution $z = \phi(t, \epsilon) \in A(\omega)$. This was shown by Malkin [7] and Bohr and Neugebauer [4] in the linear case and by Stoker [10] and, in the general case, by Bogoliubov [1].

Our main interest is $|\Re \lambda|$ small compared to $\epsilon$. We shall describe a domain, $\Omega$, in the $\lambda$-plane such that for each $\lambda \in \Omega$ the corresponding system (1) has a solution $z = \phi(t, \epsilon) \in A(\omega)$. We call $\Omega$ a nonresonance domain. We will show that $\Omega$ contains in particular $|\Re \lambda| > 1$ (this

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1 This system is derived from the second order equation $\ddot{x} + c\dot{x} + ax = f(t, x, \dot{x})$ ($f$ quasi-periodic in $t$) by the transformation $z = \dot{x} + \alpha x$ for some constant $\alpha$. 460
corresponds to the above-mentioned result of Bogoliubov) and in the remaining strip consists of a collection of closed sets each connecting the two half planes which we will call $\Omega_+$ and $\Omega_-$. (See Figure 1.) Moreover, the complement of these closed sets has small measure, independent of $\epsilon$.

We set $(\omega, k) = \sum_{\nu=1}^{m} \omega, k$, where the $k, \nu=1, \ldots, m$ are integers, and $|k| = \sum_{\nu=1}^{m} |k|$. If we assume that $g$ is analytic for $|z|, |\bar{z}| < r$, $|\text{Im} \theta| = \sum |\text{Im} \theta| < 1$ and that $|g| < 1$, then

**Theorem.** If $|\langle \omega, k \rangle| \geq e_0^{-1} |k|^{-\tau}, \ c_0 > 1, \ r > m$, then there exists $e_0 = e_0(m)$ such that for $\epsilon \leq \epsilon_0$ there exists a closed, connected set, $\Omega = \Omega(\epsilon)$ in the $\lambda$-plane such that for the corresponding system

\[ \dot{z} = \lambda z + \epsilon g(\theta, z, \bar{z}), \]
\[ \dot{\theta} = \omega \]

has a solution

\[ z = \phi(t, \epsilon) \in A(\omega). \]
The set $\Omega$ contains the half planes $\Omega_+$ and $\Omega_-$. The latter two sets are connected by infinitely many cusp-like domains bounded by curves with one point of contact. The quasi-periodic solutions are stable or unstable according as $\lambda$ lies to the left or right of the contact point.

It should be noted here that although $\Omega$ depends on $\epsilon$ and $g$ the measure of the complement is small independent of the perturbation. This implies that for most choices of $\lambda$ the system (1)' has a solution $\in A(\omega)$.

The complement of $\Omega$ is not empty even in the linear case. This can be seen as follows:

We find a solution $z$ of the linear problem by means of Fourier series. Substituting in $\dot{z} = \lambda z + \epsilon g(\theta)$ we obtain the following equations for the Fourier coefficients $z_k$ of $z$:

$$\{i(\omega, k) - \lambda\} z_k = \epsilon g_k.$$  

For $\Re \lambda = 0$, $|i(\omega, k) - \lambda|$ can be arbitrarily small since $\omega_1, \ldots, \omega_m$ are rationally independent. To prevent this we must restrict the choice of $\lambda$. If we require that $\lambda$ satisfy the inequality $|i(\omega, k) - \lambda| \geq \gamma^{-1}$ for some constant $\gamma > 1$, we find that all pure imaginary $\lambda$ are excluded. However, if we weaken the condition to

$$|i(\omega, k) - \lambda| \geq (\gamma |k|^r)^{-1}$$

where $\gamma > 1$, $r > m$, we find that the measure of the excluded set on any line parallel to the imaginary axis is proportional to $\gamma^{-1}$ and decreases as $|\Re \lambda|$ increases. Hence there are pure imaginary $\lambda$ for which $\dot{z} = \lambda z + \epsilon g(\theta)$ has a formal solution $z$ (convergence is assured if $g(\theta)$ is sufficiently differentiable).

The proof of our theorem is divided into two steps. The first and main step will consist of finding a family of curves, $\Gamma$, in the $\lambda$-plane such that for the corresponding differential equation we can

(i) construct quasi-periodic solutions belonging to $A(\omega)$,

(ii) transform the linearized equation (linearized on these solutions) into constant coefficients.

If $|\Re \lambda| > 1$ we can easily use the contraction principle on the iteration scheme

$$\begin{align*}
z_0 &= 0, \\
\dot{z}_{n+1} - \lambda z_{n+1} &= \epsilon g(\theta, z_n, z_n)
\end{align*}$$

and show convergence for $\epsilon/|\Re \lambda|$ sufficiently small. (This is essen-
Ordinary Differential Equations with Small Damping

Our main interest, however, is for $|\text{Re } \lambda|$ small. Here we need the "rapid convergence" technique of Kolmogorov [5], [6], Arnol'd [1], and Moser [8]. More precisely, we proceed as follows.

We construct a quasi-periodic transformation $z = \xi + v(\theta, \xi, \bar{\xi}, \lambda)$ taking (1)' into

$$
\begin{align*}
\dot{\xi} &= \mu \xi + \phi(\theta, \xi, \bar{\xi}, \mu) = \mu \xi + \theta(\xi^2), \\
\theta &= \omega
\end{align*}
$$

where $\mu$ satisfies

$$
|\omega, k| - j_0 \text{Im } \mu | \geq (\gamma | k | \tau)^{-1},
$$

$$
\gamma > 1, \quad \tau > m, \quad |k| \neq 0, \quad j_0 = 0, 1, 2.
$$

This provides a quasi-periodic solution $z = v(\omega t, 0, 0, \lambda) \in A(\omega)$ on a non-denumerable set of curves connecting $\Omega_+$ and $\Omega_-$. In the second step of the proof, in order to enlarge the domain we must give up the requirement that the linearized equation be transformable into constant coefficients. For every $\mu$ with $\text{Re } \mu \neq 0$ using a contraction argument we can ensure the existence of a solution $z \in A(\omega)$ if $\lambda$ is sufficiently close to the above determined curves, $\lambda = \lambda(\mu)$. It suffices to take $|\lambda - \lambda(\mu)| < c|\text{Re } \mu|^2$. This determines for each curve in $\Gamma$ a parabolic neighborhood (see Figure 1) with point of contact at $\text{Re } \lambda = 0$.

It should be noted here that the point of contact need not be on $\text{Re } \lambda = 0$. However, for reversible systems $(g(\theta, z, \bar{z}) = [-\bar{g}(-\theta, -z, -\bar{z})])$ it was shown by Moser [9] that all contact points lie on $\text{Re } \lambda = 0$.

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