A NOTE ON MINIMAL VARIETIES

JAMES SIMONS

Communicated by Eugenio Calabi, January 4, 1967

1. Introduction. In [1] Almgren considered the situation of a closed minimal variety $H$, of dimension 2 immersed in $S^3$. He observed that the second fundamental form, a real valued bilinear form on the tangent space to $H$, is in fact the real part of a holomorphic quadratic differential with respect to the conformal structure on $H$ induced by the metric inherited from its immersion in $S^3$. He used this fact to conclude that $S^2$ could not be immersed as a minimal variety in $S^3$ unless it was already totally geodesic.

It turns out that under the most general circumstances the second fundamental form of a $p$-dim minimal subvariety of an $n$-dim Riemannian manifold satisfies a natural second-order elliptic differential equation which generalizes the holomorphic condition mentioned above. In the case that the ambient manifold is $S^n$ the equation may be used to show that a closed minimal subvariety of $S^n$, of arbitrary codimension, which does not twist too much is already totally geodesic. In a sense this theorem is analogous to Bernstein's theorem for complete minimal subvarieties in $R^n$.

2. A standard operator. Let $M$ be a Riemannian manifold of dimension $n$ and $V(M)$ a $d$-dimensional vector bundle over $M$. Suppose the fibers of $V(M)$ carry a euclidean inner product and suppose there is given a connection in $V(M)$ which preserves this inner product. If $W$ is a cross-section in $V(M)$ and $x \in T(M)_m$, the tangent space to $M$ at $m$, we denote by $\nabla_x W$ the covariant derivative of $W$ in the $x$ direction. $\nabla_x W \in V(M)_m$.

Let $x, y \in T(M)_m$. We define $\nabla_x y W \in V(M)$ as follows. Let $Y$ be a vector field on $M$ which extends $y$. We then set

\begin{equation}
\nabla_x y W = \nabla_x \nabla_Y W - \nabla_{\nabla_Y W}
\end{equation}

where $\nabla_Y Y$ is ordinary covariant differentiation of a vector field on $M$ with respect to the Riemannian connection. It is easy to see that this definition is independent of the choice of $Y$.

Let $e_1, \cdot \cdot \cdot , e_n$ be an orthonormal basis of $T(M)_m$. If $W$ is a cross-section in $V(M)$ we define $\nabla^i W$ by

---

1 Prepared with partial support from NSF GP 4503.
2 All manifolds will be assumed to be orientable.

491
This definition of $\nabla^2$ is independent of the choice of frame $e_1, \ldots, e_n$. Thus, $\nabla^2$ is a second-order differential operator mapping the space of cross-sections of $V(M)$ into itself.

**Proposition 2.1.** $\nabla^2$ is an elliptic operator. If $M$ is compact we have

\begin{align*}
(2.3) & \quad \int_M \langle \nabla^2 W, Z \rangle = \int_M \langle W, \nabla^2 Z \rangle, \\
(2.4) & \quad \int_M \langle \nabla^2 W, W \rangle \leq 0, \\
(2.5) & \quad \int_M \langle \nabla^2 W, W \rangle = 0 \iff \nabla^2 W = 0
\end{align*}

$\iff W$ is covariant constant.

3. **The second fundamental form.** Let $M$ be an $n$-dimensional $C^\infty$ Riemannian manifold, $H$ a $p$-dimensional manifold, and $\Phi: H \to M$ an immersion. We consider the following vector bundles over $H$: $T(H)$ = the tangent bundle; $N(H)$ = the normal bundle; $S(H)$ = the bundle of symmetric linear transformations of $T(H)_h \to T(H)_h$; $A(H) = \text{Hom}(N(H), S(H))$. Each of these vector bundles has a natural euclidean inner product on its fibers, and each has a natural connection which preserves this inner product.

The second fundamental form $\alpha$ is a cross-section in $A(H)$. That is, for $w \in N(H)_h$, $\alpha(w): T(H)_h \to T(H)_h$ is a symmetric linear transformation. $H$ is immersed as a minimal variety if and only if for each $h \in H$ and each $w \in N(H)_h$, $\text{tr} \, \alpha(w) = 0$.

$\alpha$ gives rise to two natural linear maps at each point

\[ \tilde{\alpha}: N(H)_h \to N(H)_h; \quad \alpha: S(H)_h \to S(H)_h \]

defined as follows. Since $N(H)_h$ and $S(H)_h$ are euclidean we may define $\alpha^* =$ transpose of $\alpha$. $\alpha^*: S(H)_h \to N(H)_h$. We then set

\[ \tilde{\alpha} = \alpha^* \circ \alpha. \]

Let $f_1, \ldots, f_d$ be an orthonormal basis for $N(H)_h$, where $d = n - p$. We then set

\[ \alpha = \sum_{i=1}^{d} (\text{ad}(\alpha(f_i)))^2. \]
This definition is independent of the choice of frame \( \{ f_i \} \).

Using \( \mathcal{G} \) and \( \mathcal{A} \) we define \( \mathcal{A}(\mathcal{G}) \), a new cross-section in \( A(H) \) by

\[
\mathcal{A}(\mathcal{G}) = \mathcal{G} \circ \mathcal{A} + \mathcal{A} \circ \mathcal{G}.
\]

Let \( R \) denote the curvature tensor of \( M \). We use the convention that for \( x, y \in T(M)_m \) and orthonormal, the sectional curvature, \( k(x, y) \) of the plane section spanned by \( x \) and \( y \) satisfies \( k(x, y) = -\langle R_x y, x \rangle \). By letting \( R \) operate on \( \mathcal{G} \) we will construct a new cross-section, \( R(\mathcal{G}) \), in \( A(H) \).

For \( x, y \in T(M)_{\phi(h)} \), \( R_{x,y} : T(M)_{\phi(h)} \rightarrow T(M)_{\phi(h)} \) is a skew symmetric linear transformation. It induces:

\[
R^N_{x,y} : N(H)_h \rightarrow N(H)_h,
\]

\[
R^T_{x,y} : T(H)_h \rightarrow T(H)_h,
\]

\[
\langle R^N_{x,y} z, w \rangle = \langle R_{x,y} z, w \rangle \quad z, w \in N(H)_h,
\]

\[
\langle R^T_{x,y} z, w \rangle = \langle R_{x,y} d\phi(z), d\phi(w) \rangle \quad z, w \in T(H)_h.
\]

Then \( R^N_{x,y} \) and \( R^T_{x,y} \) are skew symmetric.

Let \( e_1, \ldots, e_p \) be a frame in \( T(H)_h \). Let \( w \in N(H)_h \) and \( x, y \in T(H)_h \).

We define the cross-section, \( R(\mathcal{G}) \), in \( A(H) \):

\[
\langle R(\mathcal{G})(w)(x), y \rangle = \sum_{i=1}^p \left\{ 2\langle \mathcal{G}(R^N_{x,y} e_i)(e_i), y \rangle + 2\langle \mathcal{G}(R^N_{w,y} e_i)(e_i), x \rangle + \langle \mathcal{G}(R^T_{x,y} e_i)(x), y \rangle - 2\langle \mathcal{G}(w)(e_i), R^T_{x,y} e_i \rangle - \langle \mathcal{G}(w)(y), R^T_{x,y} e_i \rangle \right\}.
\]

In the above expression, which is independent of the choice of \( \{ e_i \} \), we have sometimes identified points in \( T(H)_h \) with points in \( T(M)_{\phi(h)} \).

E.g., \( R^N_{x,y} = R^N_{\phi(x), \phi(y)} \).

Finally, we construct a third cross-section in \( A(H) \) which exists independently of \( \mathcal{G} \). For \( x \in T(M)_{\phi(h)} \) let \( \nabla_x(R) \) denote the standard covariant derivative of the curvature tensor. We now define \( R' \in A(H)_h \):

\[
\langle R'(w)(x), y \rangle = \sum_{i=1}^p \left\{ \langle \nabla_x(R)(e_i, y), w \rangle + \langle \nabla_y(R)(e_i, x), w \rangle + \langle \nabla_w(R)(e_i, e_i), y \rangle \right\}.
\]

**Lemma 3.1.** If \( d = n - p = 1 \), \( \overline{\mathcal{G}}(\mathcal{G}) = \| \mathcal{G} \|^p \mathcal{G} \). If \( d \geq 2 \), \( 0 \leq \langle \overline{\mathcal{G}}(\mathcal{G}), \mathcal{G} \rangle \leq \| \mathcal{G} \|^4 \).

**Lemma 3.2.** If \( M = S^n \) then \( R(\mathcal{G}) = p \mathcal{G} \) and \( R' = 0 \).
4. Minimal varieties.

**Theorem 4.1.** Let $H$ be a $C^\infty$ manifold of dimension $p$, $M$ a $C^\infty$ Riemannian manifold of dimension $n$, and $\phi: H \to M$ an immersion. Suppose the image of $H$ in $M$ is a minimal variety. Then the second fundamental form, $\alpha$, when regarded as a cross-section in the vector bundle $A(H)$ satisfies the equation:

$$(4.1) \quad \nabla^2 \alpha = -\bar{\alpha}(\alpha) + R(\alpha) + R'.$$

**Theorem 4.2.** Let $H$ be a $C^\infty$ $p$-dimensional manifold immersed in $S^n$ as a minimal variety. Then the second fundamental form, $\alpha$ satisfies the equation

$$(4.2) \quad \nabla^2 \alpha = -\bar{\alpha}(\alpha) + \rho \alpha.$$

**Corollary 4.1.** Let $H$ be a closed $p$-dimensional manifold immersed in $S^n$ as a minimal variety. Then if at each point of $H$ $\|\alpha\|^2 < \rho$, $H$ is totally geodesic, i.e., the image of $H$ in $S^n$ is the intersection of $S^n$ with a $p$-dimensional subspace of $\mathbb{R}^{n+1}$.

**Theorem 4.3.** Let $H$ be an immersed minimal variety of codimension 1 in $S^n$. Then the second fundamental form, $\alpha$, satisfies the equation

$$(4.3) \quad \nabla^2 \alpha = (n - 1 - \|\alpha\|^2)\alpha.$$

Under the hypothesis of codimension 1 Formula (4.3) may be rewritten in a form which makes it subject to more careful analysis. Let $V$ denote the unit normal vector field to $H$, chosen to make the orientation come out right. The second fundamental form, $\alpha$, may now be regarded as a real valued symmetric bilinear form $B$, defined by

$$B(x, y) = \langle \alpha(V)(x), y \rangle.$$

**Theorem 4.4.** Let $H$ be an immersed minimal variety of codimension 1 in $S^n$. Let $\overline{R}$ denote the curvature of $H$ with respect to the metric inherited from the immersion. Let $e_1, \ldots, e_{n-1}$ be a frame in $T(H)$. Then $B$ satisfies the equation

$$(4.4) \quad \nabla^2 B(x, y) = -\sum_{i=1}^{n-1} B(\overline{R}e_i, x, e_i, y) + B(e_i, \overline{R}e_ixy).$$

Equation (4.4) is interesting because both sides are defined intrinsically in terms of the geometry on $H$ inherited from the immersion. The operator on the right-hand side is almost identical to the
curvature operator on skew symmetric bilinear forms which appear as the linear piece of the Laplace-Beltrami operator. Although it is probably far from the best theorem, we can easily prove:

**Theorem 4.5.** Let $g$ denote the standard metric on $S^p$. There exists a neighborhood of $g$ in the space of nonequivalent Riemannian structure such that $S^p$ together with any metric $g'$ in this neighborhood cannot be isometrically immersed in $S^n$ as a minimal variety.

Finally, we will express Equation (4.4) as a first-order condition on $B$ and we will make the connection with holomorphic quadratic differentials mentioned in §1.

**Theorem 4.6.** Let $B$ be a field of symmetric bilinear forms on a compact Riemannian manifold, $H$. Suppose $\text{tr } B = 0$. Then $B$ satisfies (4.4) if and only if $B$ satisfies

$$(4.5) \quad \nabla_x(B)(y, z) = \nabla_y(B)(x, z), \quad \forall x, y, z \in T(H).$$

If $\dim H = 2$, $B$ satisfies (4.5) and $\text{tr } B = 0$ if and only if the form $Q(x) = B(x, x) - iB(x, j(x))$ is a holomorphic quadratic differential ($j$ being the usual 90° rotation). How to relate the dimension of the space of such forms on manifolds of higher dimension to some differential or geometric invariants seems to be a good problem.

**Bibliography**