NONTRIVIAL m-INJECTIVE BOOLEAN ALGEBRAS
DO NOT EXIST

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We adopt the notation of Sikorski [3] with the following additions. A Boolean algebra \( \mathfrak{A} \) is trivial iff it has only one element. \( \mathfrak{A} \) is m-injective iff \( \mathfrak{A} \) is an m-algebra and whenever we are given m algebras \( \mathfrak{B} \) and \( \mathfrak{C} \) with m-homomorphisms \( f, g \) of \( \mathfrak{B} \) into \( \mathfrak{A} \) and \( \mathfrak{C} \) into \( \mathfrak{A} \) respectively, and with \( g \) one-one, there is an m-homomorphism \( k \) of \( \mathfrak{C} \) into \( \mathfrak{A} \) such that \( f = k \circ g \) (\( \circ \) denotes composition of functions). Obviously every trivial Boolean algebra is m-injective for any \( m \). Halmos [1] raised the question concerning what \( \sigma \)-injective Boolean algebras look like, and Linton [2] derived interesting consequences from the assumption that nontrivial \( \sigma \)-injectives exist.

The theorem of the title follows easily from the following two lemmas, the first of which is well known, while the second has some independent interest.

**Lemma 1.** If \( \mathfrak{A} \) satisfies the m-chain condition, \( \{ A_t \}_{t \in T} \) is a set of elements of \( \mathfrak{A} \), and \( \bigcup_{t \in T} A_t \) exists, then there is a subset \( S \) of \( T \) with \( \mathfrak{S} \leq \mathfrak{m} \) such that \( \bigcup_{s \in S} A_s \) exists and equals \( \bigcup_{t \in T} A_t \).

**Proof.** Let \( \mathfrak{A} \) be a maximal set of pairwise disjoint elements of \( \mathfrak{A} \) such that for every \( B \subseteq \mathfrak{B} \) there is a \( t \in T \) such that \( B \subseteq A_t \) (such a \( \mathfrak{B} \) exists by Zorn's lemma). With every \( B \subseteq \mathfrak{B} \) one can associate an element \( t(B) \) of \( T \) such that \( B \subseteq \bigcup_{t(B)} A_t(B) \). By the m-chain condition, \( \mathfrak{S} \leq \mathfrak{m} \), and hence also \( \{ t(B) \}_{B \subseteq \mathfrak{B}} \) is m-indexed. Now \( \bigcup_{B \subseteq \mathfrak{B}} B \) exists and equals \( \bigcup_{t \in T} A_t \). For, if this is not true then, by virtue of the fact that \( B \subseteq \bigcup_{t \in T} A_t \) for each \( B \subseteq \mathfrak{B} \), it follows that there is a \( C \neq \Lambda \) such that \( B \cap C = \Lambda \) for all \( B \subseteq \mathfrak{B} \), while \( C \subseteq \bigcup_{t \in T} A_t \). Then \( C \cap A_{t_0} \neq \Lambda \) for a certain \( t_0 \in T \), and \( \mathfrak{B} \cup \{ C \cap A_{t_0} \} \) is a set properly including \( \mathfrak{B} \) with all the properties of \( \mathfrak{B} \). This contradiction shows that \( \bigcup_{B \subseteq \mathfrak{B}} B \) exists and equals \( \bigcup_{t \in T} A_t \). Obviously, then, \( \bigcup_{B \subseteq \mathfrak{B}} A_{t(B)} \) also exists and equals \( \bigcup_{t \in T} A_t \), as desired.

**Lemma 2.** For every \( m \) there is a complete Boolean algebra \( \mathfrak{A} \) such that every nontrivial \( \sigma \)-homomorphic image of \( \mathfrak{A} \) has cardinality at least \( m \).

**Proof.** Let \( \mathfrak{B} \) be a free Boolean algebra on \( m \) generators, and let \( \mathfrak{A} \) be a completion of \( \mathfrak{B} \). By [3, pp. 72, 156], \( \mathfrak{A} \) satisfies the \( \sigma \)-chain condition. Let \( I \) be a proper \( \sigma \)-ideal of \( \mathfrak{A} \). By Lemma 1, \( I \) is principal;
say \( I \) is generated by \( A \subseteq \mathcal{A} \). Then \( \mathcal{A}/I \) is isomorphic to \( \mathcal{A}|(-A) \) (see [3, pp. 30–31]); moreover, \( I \) proper implies that \( -A \neq \mathcal{A} \). But \( \mathcal{A} \) is homogeneous (see [3, pp. 106, 152]), and hence \( \mathcal{A} \) is isomorphic to \( \mathcal{A}/I \). Thus \( \mathcal{A}/I \) has at least \( m \) elements, as desired.

**Theorem.** Nontrivial \( m \)-injective Boolean algebras do not exist.

**Proof.** Suppose that \( \mathcal{A} \) is a nontrivial \( m \)-injective Boolean algebra. Let \( \mathcal{B} \) be the two-element subalgebra of \( \mathcal{A} \), and let \( \mathcal{C} \) be a complete Boolean algebra such that every nontrivial \( \sigma \)-homomorphic image of \( \mathcal{C} \) has power greater than \( \mathcal{A} \). Let \( f \) and \( g \) be the natural isomorphisms of \( \mathcal{B} \) and \( \mathcal{A} \) and \( \mathcal{A} \) into \( \mathcal{C} \) respectively. By the \( m \)-injectiveness of \( \mathcal{A} \) we obtain an \( m \)-homomorphism from \( \mathcal{C} \) onto a nontrivial subalgebra of \( \mathcal{A} \), which is impossible.

Linton has remarked to the author that this theorem can be improved to show that the category of \( m \)-algebras does not have a cogenerator.

**References**