DUALITY AND ORIENTABILITY IN BORDISM THEORIES

BY DUANE O'NEILL

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1. Introduction. A Poincaré duality theorem appears in the literature of bordism theory in several places e.g. [1], [4]. In certain $K(\pi)$-theories, i.e. classical (co)homology theories, the connection between orientability of the tangent bundle of a manifold and this duality is well known [5]. It is interesting to see how this same relationship holds in $MG$-theories and that a simultaneous proof can be given for several different $G$.

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2. Notation. Throughout this note $G_n$ will be one of $O(n)$, $SO(n)$, $U(n)$ or $SU(n)$. We let $\theta = \theta(G_n)$ be the disk bundle associated to the universal $G_n$-bundle. The Thom space, $MG_n$, is the total space of $\theta$ with the boundary collapsed to a point, the basepoint of $MG_n$. The Whitney sum of $G$-disk bundles induces the maps necessary to define the Thom spectrum $MG$ and the maps giving the (co)homology products. We will denote by $(G^*(\ ))G^*(\ )$ the (co)bordism theory associated to $MG$ as in the classical work of G. W. Whitehead [6].

Let $dn$ be the real dimension of the fiber of $\theta$. The inclusion of a fiber into the total space of $\theta$ can be thought of as a bundle map covering the inclusion of the basepoint into the classifying space for $G_n$. There is then the associated map of Thom spaces which we denote by $e_n: S^{dn} = D^{dn}/\partial D^{dn} \rightarrow MG_n$. If $f: S^qX \rightarrow MG_n$ is a map, then we denote the associated cohomology class by $(f)^G(S^qX)$. It is easy to prove using the techniques of [6] that $(e_n)$ is the identity element of $G^d(S^{dn})$ and that the identity element

$$e \in G^0(S^0) \xrightarrow{\Sigma^{dn}} (e_n) \in G^d(S^{dn})$$

where $\Sigma^{dn}$ is the iterated suspension isomorphism.

Now we consider a closed differentiable $n$-manifold $N^n$ and let $r: N \rightarrow BO(2(n+k))$ be the map classifying the stable unoriented tangent bundle of $N$. There is the sequence

$$BSU(n+k) \rightarrow BU(n+k) \rightarrow BSO(2(n+k)) \rightarrow BO(2(n+k)).$$

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Let $\tau_\theta$ be the lift of $\tau$ to the appropriate classifying space, if possible. Let $X$ be a topological space; $f: N \to X$ a map. The set of all triples $(N, \tau_\theta, f)$ forms a monoid under disjoint union. We factor by the relation: $(N, \tau_\theta, f) = 0$ iff there exists $W^{n+1}$ a differentiable manifold, a $\tau'_\theta$ for $W$ and an $f': W \to X$ such that (i) $\partial W = N$, (ii) $\tau'_\theta|\partial = \tau_\theta$, (iii) $f'|\partial = f$. It can be verified that the resulting group is isomorphic to $G_n(X)$ e.g. [1], [2].

3. Thom classes. Now we define classes $V_n \in \tilde{C}^{dn}(MG_n)$ by $V_n = (I_n)$ where $I_n: MG_n \to MG_n$ is the identity map and we state the

**Theorem 1.** $V_n$ is an $MG$-orientation class for $\theta$.

**Proof.** We must show that $e^*_n((V_n)) \in \tilde{C}^{dn}(S^d)$ is the suspension of $e \in \tilde{C}^0(S^0)$. But it is obvious that $e^*_n(V_n) = (e_n \circ I_n) = (e_n)$.

It follows from [3] that the cohomology theory $G^*$ has a Thom isomorphism and a Gysin sequence for $G$-plane bundles.

In the remainder of this note, let $A = Z_2$ or $Z$ as appropriate. There exist Thom classes $u_n \in \tilde{B}^{dn}(MG_n, A)$ i.e. $\theta$ is $K(A)$-orientable. Let $h_n: (MG_n, *) \to (K(A, dn), *)$ be a representative for the homotopy class $u_n$ for every $n$. We obtain (if $d = 2$, we must extend the collection of maps by some suspensions) a spectral map $h: MG \to K(A)$. We notice that the preceding choice of a Thom class in $G^*$-theory implies that $h$ maps $v_n \to u_n$. Hence, we see the

**Lemma.** $h$ is an isomorphism on the coefficients $\tilde{C}^n(S^n)$.

**Remark.** In [2] Conner and Floyd introduced another spectral map of this type which they call $\beta: U^* \to K^*$. Since the definitions of $\beta$ and $h$ depend on the Thom classes of the range theory, $V_n$ always maps to the $n$th Thom class. A first Chern class is also defined in [2]. It can be shown that the definition of [2] coincides with the usual definition of first Chern class as $\theta^* \circ j^*(V_1)$, where $\theta$ is the zero section of the bundle and $j$ the map collapsing the total space of the bundle to its Thom space.

4. Manifolds. In the remainder of this note, let $M$ be a differentiable manifold of real dimension $m$, whose stable tangent bundle is a $G_l$-bundle. We choose representatives $s_n: S^n \to K(A, n)$ for the homotopy class of the generator of $\tilde{B}^n(S^n, A)$ and obtain a spectral map $s: S \to K(A)$.

The diagram
commutes and induces the corresponding commutative diagram for the associated cohomology theories. Following [6, p. 271] we write the fundamental cocycle of $H^m(M; A)$ as $s(j) = \tilde{Z}(M)$ where $j: M \to S^m$ is a map collapsing everything outside a coordinate disk to the basepoint. We call the class $\tilde{\sigma}(M) = e((j)) \in G^m(M)$ the fundamental cocycle as well. Note that $h(\tilde{\sigma}(M)) = \tilde{Z}(M)$. Recall the definition of [6] that $M$ is $MG$ orientable iff there exists a $t \in G_m(M)$ so that $(\tilde{\sigma}(M), t) = e \in G_0(S^0)$.

**Theorem 2.** $M$ a differentiable manifold with $G_1$ the group of the stable tangent bundle implies that $M$ is $MG$ orientable.

**Corollary.** If $M$ as above then $M$ has Poincaré duality with respect to $MG$ (from [6]).

**Proof of Theorem 2.** Consider $\sigma(M) = [M, \tau_G, id_M] \in G_m(M)$. One can show that $h(\sigma(M)) = Z(M)$ the fundamental cycle in $H^m(M; A)$ by first showing that $h$ can be identified with an edge homomorphism in a spectral sequence relating $G_*$ and $H_*$ (see [1, p. 17]). Then it is not difficult to calculate the value of $\sigma(M)$.

And finally we have the commutative diagram

\[
\begin{array}{ccc}
G^m(M) \otimes G_m(M) & \xrightarrow{\langle , \rangle} & G_0(pt) \\
\downarrow h \otimes h & & \downarrow h \\
H^m(M; A) \otimes H_m(M; A) & \xrightarrow{\langle , \rangle} & H_0(pt)
\end{array}
\]

where the $h$ on the right is an isomorphism.

**References**