THE DIFFEOMORPHISM GROUP OF A COMPACT RIEHMANN SURFACE

BY C. J. EARLE¹ AND J. EELLS

Communicated by S. Smale, February 6, 1967

1. Introduction. In this note we announce two theorems. The first describes the homotopy type of the topological group $\mathcal{D}(X)$ of diffeomorphisms ($= C^\infty$-diffeomorphisms) of a compact oriented surface $X$ without boundary. The second, of which the first is a corollary, gives a fundamental relation among $\mathcal{D}(X)$, the space of complex structures on $X$, and the Teichmüller space $T(X)$ of $X$. We make essential use of the theory of quasiconformal mappings and Teichmüller spaces developed by Ahlfors and Bers \[3\], \[6\], and the theory of fibrations of function spaces. Our results confirm a conjecture of Grothendieck \[7\], p. 7-09, relating the homotopy of $\mathcal{D}(X)$ and $T(X)$.

2. The theorems. The surface $X$ has a unique (up to equivalence) $C^\infty$-differential structure. Let $\mathcal{D}(X)$ denote the group of orientation preserving diffeomorphisms. With the $C^\infty$-topology (uniform convergence of all differentials) $\mathcal{D}(X)$ is a metrizable topological group \[8\]. We let $\mathcal{D}_0(X; x_1, \ldots, x_n)$ denote the subgroup of $\mathcal{D}(X)$ consisting of those diffeomorphisms $f$ which are homotopic to the identity and satisfy $f(x_i) = x_i$ ($1 \leq i \leq n$), where $x_1, \ldots, x_n$ are distinct points of $X$. This second condition is fulfilled vacuously if $n = 0$.

**Theorem 1.** Let $g$ denote the genus of $X$.

(a) If $g = 0$, then $\mathcal{D}_0(X; x_1, x_2, x_3)$ is contractible. Furthermore, $\mathcal{D}(X)$ is homeomorphic to $G \times \mathcal{D}_0(X; x_1, x_2, x_3)$, where $G$ is the group of conformal automorphisms of the Riemann sphere.

(b) If $g = 1$, then $\mathcal{D}_0(X; x_1)$ is contractible. Furthermore, $\mathcal{D}_0(X)$ is homeomorphic to $G \times \mathcal{D}_0(X; x_1)$, where now $G$ is the identity component of the group of conformal automorphisms of the torus.

(c) If $g \geq 2$, then $\mathcal{D}_0(X)$ is contractible.

**Corollary.** In all cases $\mathcal{D}_0(X)$ is the identity component of $\mathcal{D}(X)$.

**Remark 1.** Part (a) is equivalent to the theorem of Smale \[9\] asserting that the rotation group $SO(3)$ is a strong deformation retract of $\mathcal{D}(S^2)$. Our proof is entirely different from Smale's.

**Remark 2.** A concept of differentiability has recently been de-

¹ Research partially supported by NSF Grant GP6145.
veloped by J. A. Leslie, relative to which he has shown that the group \( \mathcal{D}(X) \)—for any compact differential manifold—is a Lie group \[8\].

Next, let \( M(X) \) denote the \( C^\infty \)-complex structures on \( X \); with the \( C^\infty \)-topology this is a convex open subset of a Fréchet space. Any subgroup of \( \mathcal{D}(X) \) operates (on the right) on \( M(X) \) in the obvious way. If \( g \geq 1 \) let \( T(X) \) denote the Teichmüller space of \( X \). For the definition of that space we refer to [2], [6, Lecture 1]; however, Theorem 2 below provides a new characterization.

**Theorem 2.** Let \( g \) denote the genus of \( X \).

(a) If \( g = 0 \), then \( \mathcal{D}_0(X; x_1, x_2, x_3) \) is homeomorphic to \( M(X) \).

(b) If \( g = 1 \), then \( \mathcal{D}_0(X; x_1) \) operates principally (i.e., continuously, freely, properly, and with local sections) on \( M(X) \). The quotient space is (homeomorphic to) \( T(X) \).

(c) If \( g \geq 2 \), then \( \mathcal{D}_0(X) \) operates principally on \( M(X) \). The quotient space is (homeomorphic to) \( T(X) \).

3. **On the proof of Theorem 1.** We proceed to indicate how Theorem 1 is derived from Theorem 2.

First of all, \( M(X) \) is always contractible. Therefore Theorem 2a easily implies Theorem 1a. Secondly, the quotient map \( M(X) \rightarrow T(X) \) defines a principal fibre bundle. A fundamental theorem of Teichmüller (see [1], [5] for efficient proofs) asserts that \( T(X) \) is a finite dimensional cell. Thus from the homotopy sequence of our fibration we conclude that all the homotopy groups of the structural groups \( \mathcal{D}_0(X; x_1) \) and \( \mathcal{D}_0(X) \) vanish. Finally, these structural groups are metrizable manifolds modeled on Fréchet spaces; therefore, they are absolute neighborhood retracts. By a theorem of J. H. C. Whitehead the vanishing of their homotopy groups implies their contractibility. Theorem 1 follows.

4. **On the proof of Theorem 2.** For simplicity of exposition we consider only the cases \( g \geq 2 \). We represent \( X \) as the quotient of the upper half plane \( U \) by a Fuchsian group \( \Gamma \) operating freely on \( U \). The \( C^\infty \) complex structures (\( = C^\infty \) Beltrami differentials [2]) on \( X \) are represented by the \( C^\infty \) complex valued functions \( \mu \) on \( U \) satisfying

\[
\mu(\gamma z)\gamma'(z)/\gamma'(z) = \mu(z)
\]

and \( \max \{ |\mu(z)| : z \in U \} < 1 \). Each such function \( \mu \) determines uniquely a diffeomorphism \( f : U \rightarrow U \) such that

\[
\mu(z) = \mu_f(z) = f_*(z)/f_*(z),
\]

and \( f \) leaves the points 0, 1, \( \infty \) fixed.
The group $\mathcal{D}_0(X)$ is identified with the group of diffeomorphisms $h: U \to U$ which commute with all elements of $\Gamma$. The action of $\mathcal{D}_0(X)$ on $M(X)$ is given by

$$\mu_f \cdot h = \mu_{fh}.$$ 

This action is principal, and the quotient $M(X)/\mathcal{D}_0(X)$ is homeomorphic to $T(X)$. For the verification we first study the dependence on $\mu$ of the solutions of Beltrami's equation $f_b = \mu f_z$. Next we verify that $M(X)/\mathcal{D}_0(X)$ maps bijectively and continuously onto $T(X)$. Finally, a theorem of Ahlfors and Weill [4] provides us with local sections from $T(X)$ into $M(X)$.

REFERENCES


CORNELL UNIVERSITY AND
CHURCHILL COLLEGE, CAMBRIDGE