RESIDUALLY FINITE ONE-RELATOR GROUPS

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Introduction. It seems to be commonly believed that the presence of elements of finite order in a group with a single defining relation is a complicating rather than a simplifying factor. This note is in support of the opposite point of view, lending respectability to the

CONJECTURE A. Every group with a single defining relation with non-trivial elements of finite order is residually finite.

In order to put our results in their proper setting let us define \( \langle l, m \rangle \) to be the group generated by \( a \) and \( b \) subject to the single defining relation \( a^{-1}b^l ab^m = 1 \):

\[
\langle l, m \rangle = (a, b; a^{-1}b^l ab^m = 1).
\]

Adding a third parameter we define \( \langle l, m; t \rangle \) = \( (a, b; (a^{-1}b^l ab^m)^t = 1) \).

Let \( \mathcal{E} \) be the class of those groups \( \langle l, m \rangle \) satisfying \( |l| \neq 1 \neq |m|, lm \neq 0 \), and \( l \) and \( m \) relatively prime. Furthermore, let \( \mathcal{M} \) be the class of these groups \( \langle l, m; t \rangle \) satisfying the conditions imposed above on \( l \) and \( m \), and in addition the extra two conditions \( t > 1 \), and \( l, m \) and \( t \) relatively prime in pairs. The point of our initial remark is that \( \mathcal{M} \) looks more complicated than \( \mathcal{E} \). Actually \( \mathcal{E} \) is quite a nasty class of groups. Indeed the main result of [1] is that every group in \( \mathcal{E} \) is isomorphic to one of its proper factor groups, i.e. nonhopfian. Since finitely generated residually finite groups are hopfian (A. I. Mal'cev [2]) no group in \( \mathcal{E} \) is residually finite. Our contribution to Conjecture A is that the groups in \( \mathcal{M} \) are residually finite.

THEOREM 1. Every group in the class \( \mathcal{M} \) is residually finite.

In fact even more is true.

THEOREM 2. If \( l, m, t \) are relatively prime in pairs \((l \neq 0 \neq m)\) and if \( t \) is a power of a prime \( p \) \((t \neq 1)\) then the group \( \langle l, m; t \rangle \) is residually a finite \( p \)-group.

Conjecture A seems difficult. A somewhat easier related conjecture is

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Conjecture A. Every finitely generated group with a single defining relation with nontrivial elements of finite order is hopfian.

The theory of groups with a single defining relation has been developed sufficiently for us to be able to prove

Theorem 3. Let $G$ be a group with a single defining relation and let $T$ be the subgroup of $G$ generated by the elements of finite order. If $G/T$ is hopfian, so is $G$.

The existence of the nonhopfian group $(2, 3)$ together with Theorem 1 show that the converse of Theorem 3 is false. This underlines to some extent the difficulties involved in the proof of Theorem 1.

Remarks on the proofs. The proof of Theorem 1 goes as follows. Suppose $G \subseteq \mathfrak{A}$. Thus

$$G = (a, b; (a^{-1}b^t a)^t = 1).$$

We observe that if $N$ is the normal closure of $b$ in $G$ then $G/N$ is infinite cyclic. Our procedure is to prove that $N$ is residually finite. Since an extension of a residually finite group by another residually finite group need not be residually finite we have to establish that $N$ is residually finite in such a way that we are able to deduce the residual finiteness of $G$. To establish the results we need about $N$ we have to obtain sufficient information about certain one-relator subgroups from which $N$ is constructed. This information is contained in the following lemmas.

Lemma 1. The groups

$$(a, b; (a^tb^m)^t = 1) \quad (t > 1)$$

contain a normal subgroup of finite index which is residually free.

Lemma 2. The groups

$$(a, b; (a^tb^m)^t = 1) \quad (t > 1)$$

are residually finite $p$-groups if $t$ is a power of the prime $p$.

Both Lemma 1 and Lemma 2 make use of the Reidemeister-Schreier procedure for finding generators and defining relations for a subgroup of a group given by generators and defining relations (see [3, p. 86]) as well as the main results of [4] and [5] on the residual properties of certain generalized free products.

The proof of Theorem 2 involves a refinement of the proof of Theorem 1 and an old theorem of P. Hall, namely that an automorphism of
a finite $p$-group $P$ which induces an automorphism of $p$-power order on $P$ modulo its Frattini subgroup is itself of $p$-power order (see e.g. [6, p. 178]).

Finally the proof of Theorem 3 depends on the known structure of $T$ [7] and the fact that in a one-relator group every pair of elements of maximal finite order are conjugate [8].

**Extension of results.** Theorem 1 can be extended to certain groups with a single defining relation on more than two generators. At the present time I am unable to relax the conditions on $l$, $m$ and $t$ to $t>1$. But it is certainly likely that $\langle l, m; t \rangle$ ($t>1$) is residually finite. This can probably be proved by similar arguments to those used in the proof of Theorem 1. A proof of Conjecture A, however, at this time, seems out of reach.

**References**

2. A. I. Mal'cev, *On isomorphic representations of infinite groups by matrices*, Mat. Sb. 8 (1940), 405–422.