GLOBAL CONTINUOUS SOLUTIONS OF HYPERBOLIC SYSTEMS OF QUASI-LINEAR EQUATIONS

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Recently there have appeared a number of results on global solutions of the Cauchy problem for hyperbolic systems of quasi-linear equations [2], [3], [4], [5]. These solutions are in general discontinuous. In certain cases, however, such as the interaction of two rarefaction waves in gas dynamics, it is known that the Cauchy problem has a global continuous solution [1, pp. 191–197]. In this announcement we outline a proof that a global continuous solution exists and is unique for a two-dimensional system provided the Riemann invariants associated with the initial data satisfy certain monotonicity and continuity conditions.

Let \( \lambda^+(r, s) \), \( \lambda^-(r, s) \) be \( C^1 \) real-valued functions on a domain \( D \subset R_2 \), with

\[
\lambda^+(r, s) > \lambda^-(r, s), \quad \frac{\partial \lambda^+(r, s)}{\partial r} > 0, \quad \frac{\partial \lambda^-(r, s)}{\partial s} > 0
\]

for \( (r, s) \in D \). Consider the two-dimensional system of quasi-linear equations in Riemann invariant form

\[
\begin{align*}
r_t + \lambda^+(r, s)r_x &= 0, \\
s_t + \lambda^-(r, s)s_x &= 0
\end{align*}
\]

where \( r(t, x) \) and \( s(t, x) \) are real-valued functions of two scalar variables. We seek a solution of the Cauchy problem in the halfplane \( \{(t, x) \in R_2 : t \geq 0\} \) with initial conditions

\[
\begin{align*}
r(0, x) &= r_0(x), \\
s(0, x) &= s_0(x), \\
-\infty &< x < +\infty.
\end{align*}
\]

Let \( G_T = \{(t, x) \in R_2 : 0 \leq t < T\} \) for \( 0 < T \leq +\infty \). A pair of Lipschitz continuous functions \( (r(t, x), s(t, x)) \), \( (t, x) \in G_T \), is called a Lipschitz continuous solution of the Cauchy problem (2), (3) if \( r(t, x) \) is constant on the integral curves

\[
x'(t) = \lambda^+(r(t, x), s(t, x)),
\]

\( s(t, x) \) is constant on the integral curves

\[
x'(t) = \lambda^-(r(t, x), s(t, x)),
\]

\footnote{These results are part of the author’s Ph.D. thesis written under Joel A. Smoller at the University of Michigan.}
and $r(0, x), s(0, x)$ satisfy the initial conditions (3). The pair $(r(t, x), s(t, x))$ is called a global Lipschitz continuous solution of (2), (3) if the functions are defined and Lipschitz continuous on $G_\infty$.

**Theorem 1.** If $r^0(x), s^0(x), -\infty < x < +\infty$, are bounded, Lipschitz continuous, and nondecreasing, satisfying

$$[r^0(-\infty), r^0(+\infty)] \times [s^0(-\infty), s^0(+\infty)] \subset D,$$

then the Cauchy problem (2), (3) with initial functions $r^0(x), s^0(x), -\infty < x < +\infty$, has a global Lipschitz continuous solution which takes its values in the rectangle (6).

**Outline of Proof for Theorem 1.** For each finite subset $A$ of $R_1$ we construct an approximate solution $(r(t, x; A), s(t, x; A)), (t, x) \in G_\infty$, with the property that $r(t, x; A)$ is constant on curves of the form (4) and $s(t, x; A)$ is constant on a finite number of curves of the form (5). Using condition (1) and the assumed properties of the initial functions, we show that $r(t, x; A)$ is Lipschitz continuous in $G_\infty$ with Lipschitz constant independent of $t$, $x$, and $A$. If $\{B_n : n = 1, 2, \ldots\}$ is an increasing sequence of finite sets whose union is dense in $R_1$, then by the Ascoli theorem the sequence of functions $r(t, x; B_n)$ contains a subsequence converging to a Lipschitz continuous function $r(t, x)$. Having this function, we construct $s(t, x), (t, x) \in G_\infty$, so that the pair $(r(t, x), s(t, x)), (t, x) \in G_\infty$, is a global Lipschitz continuous solution of (2), (3).

We have also obtained the following result regarding the dependence of Lipschitz continuous solutions on initial data.

**Theorem 2.** Let $r_i^0(x), s_i^0(x), -\infty < x < +\infty, i = 1, 2$, be bounded real-valued functions with

$$a_i \leq r_i^0(x) \leq b_i, \quad c_i \leq s_i^0(x) \leq d_i, \quad -\infty < x < +\infty,$$

and suppose that $[a_i, b_i] \times [c_i, d_i] \subset D, i = 1, 2$. Let

$$m = \sup_{-\infty < x < +\infty} \left( | r_i^0(x) - r_i^0(x) | + | s_i^0(x) - s_i^0(x) | \right).$$

If $(r_i(t, x), s_i(t, x)), (t, x) \in G_T$, is a Lipschitz continuous solution of the Cauchy problem for the equations (2) with initial vector $(r_i^0(x), s_i^0(x)), -\infty < x < +\infty, i = 1, 2$, then there is a constant $L(T)$ such that

$$\sup_{(t, x) \in G_T} \left( | r_1(t, x) - r_2(t, x) | + | s_1(t, x) - s_2(t, x) | \right) \leq mL(T).$$
It follows easily from this that Lipschitz continuous solutions are unique.

Theorems 1 and 2, together with results of Lax [3], can be applied to the conservation law

\[ u_t + (\phi(v))_x = 0, \quad v_t - u_x = 0 \]

to yield the following corollaries.

**Corollary 1.** Let \( \phi(v) \in C^2 \) on the halfline \( (v > 0) \) with \( \phi'(v) < 0, \phi''(v) > 0 \) and \( \int_1^\infty [-\phi'(v)]^{1/2} dv = \infty \). If the functions \( u^0(x), v^0(x) \), \( -\infty < x < +\infty \), are bounded and Lipschitz continuous, with \( v^0(x) \) positive and bounded away from 0, and satisfy

\[ \int_{v^0(x_1)}^{v^0(x_2)} [-\phi'(v)]^{1/2} dv \geq 0 \]

for \( x_2 > x_1 \), then the Cauchy problem for the equations (7) with initial vector \( (u^0(x), v^0(x)) \) has a unique Lipschitz continuous weak solution.

**Corollary 2.** Let \( \phi(v) \) be as in Corollary 1, and let \( u^0(x), v^0(x), -\infty < x < +\infty \), be bounded and piecewise constant real-valued functions, with \( v^0(x) \) positive and bounded away from 0, which satisfy (8). If the set \( A \) of discontinuities of the vector function \( (u^0(x), v^0(x)) \) has the property

\[ \inf\{ | a - b | : a, b \in A, a \neq b \} > 0, \]

then the Cauchy problem for the system (7) with initial vector \( (u^0(x), v^0(x)) \) has a solution which is Lipschitz continuous in each of the sets \( \{ (t, x) \in G_\infty : t > t_0 \} \), for \( t_0 > 0 \).

Corollary 2 provides a solution for the interaction of simple waves centered on the line \( (t = 0) \).

**References**


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