EMBEDDING PROJECTIVE SPACES

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1. Haefliger reduced the question of embedding manifolds in the Euclidian space $R^n$ to a homotopy problem in $[6]$. Since then it has been of some interest to find examples of $n$-manifolds which embed in $R^{2n-k}$ for a given $k$. In particular great effort has been spent studying embeddings of the various projective spaces. However, the $k$ that were thus obtained were in no cases larger than 5 or 6 (see for example [7], [8], [9]). Our purpose in this note is to indicate the proofs of the theorems that follow.

**Theorem 1.** Let $n = 7(8)$; then $RP^n$ (real $n$-dimensional projective space) embeds in $R^{2n-k}$ where $k \geq 2 \lceil \log_2(\alpha(n)) \rceil - 1$. (Here $\alpha(n)$ is the number of ones in the dyadic expansion of $n$.)

**Theorem 2.** If $n$ is odd and $\alpha(n)$ is greater than $4+2^t$, then $CP^n$ (complex projective space) embeds in $R^{4n-k}$ with $k \geq 3 + i$.

**Theorem 3.** If $\alpha(n) \geq 11 + 2^t$ then $QP^n$ (quaternionic projective space) embeds in $R^{8n-k}$ where $k \geq 5 + i$.

The detailed proof of Theorem 1 appears in [5] so in the sequel we will concentrate on giving those modifications which must be made in [5] so as to prove Theorems 2 and 3.

2. **A key lemma.** Let $M^n$ immerse in $R^{2n-r}$ and set $k(n) = 8s + 2^t - 1$ (where $n+1 = (2^{4s+t})c$ with $c$ odd and $0 \leq t \leq 3$). Then for $n \geq 3$ we have:

**Lemma 2.1.** (a) If $n$ is odd there are exactly two isotopy classes of immersions $M^n \subset R^{2n}$. One contains an embedding and the other an immersion with a single double point as its only singularity, but both normal bundles have $k$ independent cross-sections where $k = \min (r, k(n))$.

(b) If $n$ is even and $M^n$ orientable then there are $Z$ isotopy classes of immersions $M^n \subset R^{2n}$ only one of which contains an embedding. The only immersion with a normal field is the embedding, hence the embedding has $r$ normal fields.

**Remark.** Part b is false for nonorientable manifolds for all $n$ [4].

**Proof.** Part a follows from Whitney's well known results [10] on embeddings and immersions in $R^{2n}$, and a careful study of how one

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changes the number of double points in an immersion. The details are in [5]. To prove part b note that, according to Hirsch [1], the isotopy classes of immersions are in 1-1 correspondence with the homotopy classes of cross-sections of the stable \((n+1)\)-dimensional normal bundle. If \(n\) is even the homotopy classes of \(n\)-plane bundles stably equivalent to the stable normal bundle are classified by \(\chi/2\) (where \(\chi\) is the Euler class of the bundle). But in this case \(\chi\) is the obstruction to finding a cross-section. Finally, we note that the normal bundle to an embedding \(M^n \subset \mathbb{R}^{2n}\) must have Euler class equal to zero [3].

For \(f: M^n \subset \mathbb{R}^{n+k}\) we denote by \(\eta_f\) the normal bundle associated to \(f\).

**Corollary 2.2.** If \(n\) is even, \(M^n\) compact and orientable, and if \(v\) is a subbundle of \(\eta_f\) for some immersion \(f: M^n \subset \mathbb{R}^{2n-k}\) with \(k > 0\), then \(\eta_g\) where \(g\) is the embedding \(g: M^n \subset \mathbb{R}^{2n}\) also contains \(v\) as a subbundle.

3. **Embedding bundles over projective spaces.** Using Corollary 2.2 and the immersion results of [2] we can prove:

**Theorem 3.1.** (a) If \(2p < \alpha(n) - \alpha(p+1) - 3\), then \(\eta_{CP^n \subset OP^n}\) embeds in \(\mathbb{R}^{2q}\) where \(n = p+q+1\),
(b) If \(4p < \alpha(n) - \alpha(p+1) - 10\), then \(\eta_{QP^n \subset OP^n}\) embeds in \(\mathbb{R}^{2q}\) where \(n = p+q+1\).

The proof follows closely the arguments of §3 of [5], and in particular the argument following the proof of Lemma 3.2.

**Theorem 3.2.** (a) If \(n\) is odd then \(CP^n \subset \mathbb{R}^{4n}\) with \(\alpha(n)\) trivial sections.
(b) If \(\alpha(n) > 3\), then \(QP^n \subset \mathbb{R}^{8n}\) with \(\alpha(n) - 3\) sections.

This follows directly from the immersion results of [2] together with 2.2.

4. **Double mapping cylinders and the main theorems.** Suppose we have spaces \(X\), \(Y\), and \(Z\) and maps

\[f: Y \to Z, \quad g: Y \to X\]

then the double mapping cylinder \(M(f, g)\) is obtained from the disjoint union \(X \cup I \times Y \cup Z\) by identifying a point \((0, y)\) in \(I \times Y\) with \(f(y)\) in \(Z\) and \((1, y)\) with \(g(y)\) in \(X\). The usual mapping cylinder is obtained by setting \(X = Y\) and \(g = \text{id}\). We denote it by \(M(f)\).

Let \(FP^n\) represent either \(CP^n\) or \(QP^n\). Let \(FP^p\) be embedded in \(FP^n\) as the set of points whose last \(p+1\) homogeneous coordinates are zero (where \(n = p+q+1\)). Embed \(FP^p\) in \(FP^n\) as the set of points whose first \(q+1\) coordinates equal zero. Finally, set \(E_{p,q}\) equal to the set of points with (normalized) homogeneous coordinates \(\langle x_1, \cdots,\rangle\).
where $\sum x_i x_i = \sum y_j y_j = 1/2$. There are evident projections $\pi_1: E_{p,q} \to FP^p$, $\pi_2: E_{p,q} \to FP^q$, and we have

**Lemma 4.1.** (a) $M(\pi_1) = \eta_{FP^p} \subset FP^n$,
(b) $N(\pi_2) = \eta_{FP^q} \subset FP^n$,
(c) $M(\pi_1, \pi_2) = FP^n$.

Now, when we have spaces given as double mapping cylinders, we can use the following theorem to obtain embeddings.

**Theorem 4.2.** Retaining the previous notation let $X$ be a compact, differentiable, $n$-dimensional manifold and assume we have maps $h, T$ so that

(i) $h: X \subset \mathbb{R}^1$ with $\eta_h = k\varepsilon \oplus \bar{\eta}$ (where $\bar{\eta}$ is some subbundle of $\eta_h$ and $\varepsilon$ is the trivial line bundle),
(ii) $T: Z \subset \mathbb{R}^m$ is a topological embedding,
(iii) there is a topological embedding $S: M(f) \to \mathbb{R}^k \times \mathbb{R}^m$ so that $S$ restricted to $Z$ is $T$, then there is a topological embedding of $M(f, g)$ in $\mathbb{R}^{n+m+1}$.

The proof is contained in [5]; it is similar to the proof of Theorem 1.2 of [9].

**Remark.** When $M(f, g)$ is a manifold and we are in the metastable range then Haefliger's theorem [6] shows that we can assume the embedding is differentiable.

Now, using 4.1, 3.1 and 3.2 it is easy to complete the proofs of Theorems 2 and 3 exactly in the manner Theorem 1 is proved in [5].

**References**


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