WIENER-HOPF OPERATORS AND ABSOLUTELY CONTINUOUS SPECTRA

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Communicated by P. R. Halmos, May 10, 1967

If \( A \) is a selfadjoint operator on a Hilbert space \( \mathcal{H} \) with spectral resolution \( A = \int \lambda dE_\lambda \), it is known that the set of elements \( x \) in \( \mathcal{H} \) for which \( \|E_\lambda x\|^2 \) is an absolutely continuous function of \( \lambda \) is a subspace, \( \mathcal{H}_a(A) \), reducing \( A \); cf. Halmos [1, p. 104]. In case \( \mathcal{H}_a(A) = \mathcal{H} \), \( A \) is said to be absolutely continuous. The following was proved in Putnam [4]; see also [5] and will be stated as a

**LEMMA.** Let \( T \) be a bounded operator on a Hilbert space \( \mathcal{H} \) and let

\[
T^*T - TT^* = C, \quad C \geq 0.
\]

If \( A = T + T^* \), then \( \mathcal{H}_a(A) \supseteq \mathcal{M}_T \), where \( \mathcal{M}_T \) is the least subspace of \( \mathcal{H} \) reducing \( T \) (that is, invariant under \( T \) and \( T^* \)) and containing the range of \( C \).

The above will be used to give a short proof of the absolute continuity of certain bounded selfadjoint Wiener-Hopf operators on \( L^2(0, \infty) \). For an extensive account of Wiener-Hopf operators on the half-line see Krein [2].

Let \( k(t) \), for \(- \infty < t < \infty \), satisfy

\[
k \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty) \quad \text{and} \quad k(-t) = k(t).
\]

Then the operator \( T \) on \( \mathcal{H} = L^2(0, \infty) \) defined by

\[
(Tf)(t) = \int_0^t k(s-t)f(s)ds, \quad 0 \leq t < \infty,
\]

is bounded. (In fact, the hypothesis \( k \in L^1(-\infty, \infty) \) alone implies the boundedness of \( T \), even \( \|T\| \leq \int -\infty \|k(t)\|dt \); cf. Krein [2, pp. 201–202].) The adjoint \( T^* \), which is given by

\[
(T^*f)(t) = \int_t^\infty k(s-t)f(s)ds,
\]

and the selfadjoint operator \( A = T + T^* \), where

\[
(Af)(t) = \int_0^\infty k(s-t)f(s)ds,
\]

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1 This work was supported by a National Science Foundation research grant.
are of course also bounded on $\mathcal{S} = L^2(0, \infty)$. There will be proved the following

**Theorem.** Let $k(t)$ satisfy (2) and suppose that

$$K(\lambda) = \int_{-\infty}^{\infty} k(t)e^{-\lambda t} dt \neq 0 \text{ a.e.,} \quad -\infty < \lambda < \infty. \quad (6)$$

Then the bounded self-adjoint operator $A$ of (5) is absolutely continuous.

**Proof.** A calculation similar to that in Putnam [3, p. 517], shows that, for $f \in \mathcal{S}$, $\|Tf\|^2 - \|T^*f\|^2 = \|Bf\|^2$, where $T$ is defined in (3) and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(t + s)f(s)ds, \quad (7)$$

so that (1) holds with $C = B^*B$. It will be shown that the set $\mathbb{M}_T$ of the Lemma is, in the present case, the entire space $\mathcal{S} = L^2(0, \infty)$, and hence the absolute continuity of $A$ will follow.

For $f \in L^2(0, \infty)$, define the Fourier transform $\hat{f}(\lambda)$ and the functions $F_+(\lambda)$ and $F_-(\lambda)$ by

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) dt \equiv F_-(\lambda) \quad (8)$$

and

$$F_+(\lambda) = \int_{0}^{\infty} e^{\lambda t} f(t) dt. \quad (9)$$

The space of elements $F_+$ is a subspace of $L^2(-\infty, \infty)$ and will be denoted by $R_+$; similarly, the space of elements $F_-$ will be denoted by $R_- (= R_+)$. (It is clear that $R_+ [R_-]$ can be regarded as the space of Fourier transforms of functions in $L^2(-\infty, \infty)$ which are 0 on the left [right] half-line. Since orthogonality is preserved under Fourier transforms it follows in particular that $R_+ \perp R_-$)

If $f \in L^2(0, \infty)$ then

$$(Tf)^\vee(\lambda) = \int_{0}^{\infty} e^{-\lambda t} \left[ \int_{0}^{t} k(s - t)f(s) ds \right] dt$$

and hence, on inverting the order of integration, $(Tf)^\vee(\lambda) = K_+(\lambda)f(\lambda)$, where $K_+(\lambda)$ is defined by

$$K_+(\lambda) = \int_{0}^{\infty} e^{\lambda t} k(t) dt. \quad (10)$$
(It may be noted that \( f_k(s-t)f(s)ds \) is the convolution of \( k \) and \( f \) on \( 0 \leq t < \infty \)). More generally, an iteration shows that

\[
(T^n f)(\lambda) = K^n_{ij} f(\lambda), \quad n = 0, 1, 2, \ldots.
\]

Let \( g \perp R_{B^*B} \), so that \( Bg = 0 \), that is \( \int_0^\infty k(t+s)g(s)ds \) is the 0 element of \( \mathcal{S} = L^2(0, \infty) \). Since, by (2), \( k(t+s) \) belongs to \( L^2(0, \infty) \) for any fixed \( t \), it follows that the last integral is a continuous function of \( t \) on \( 0 \leq t < \infty \) and that

\[
m_c \in R_{B^*B}, \quad \text{where} \quad m_c(t) = \bar{k}(t+c) \quad \text{and} \quad c \geq 0.
\]

It is readily verified that

\[
m_c(\lambda) = e^{\bar{\iota}c} \int_0^\infty e^{-\bar{\iota}t\bar{k}(t)}dt.
\]

Also, in view of the definition of \( K(\lambda) \) in (6), one has

\[
K(\lambda) = K_+(\lambda) + K_-(\lambda),
\]

where \( K_+(\lambda) \) is defined in (10).

In order to prove that \( \mathcal{S}_+(A) = \mathcal{S}_+ = L^2(0, \infty) \), it is sufficient, as noted above, to prove that \( M_T = \mathcal{S} \). Now, if \( M_T \neq \mathcal{S} \), then there exists a function \( q \in \mathcal{S} \) such that \( q \neq 0 \) and \( q \perp M_T \). Let \( Q = Q(\lambda) \) denote the Fourier transform of \( q \), so that

\[
Q(\lambda) = \int_0^\infty e^{-\lambda t}q(t)dt \quad (\in \mathcal{R}_-).
\]

In view of (12), it follows from the relation \( q \perp M_T \) and the fact that orthogonality is preserved under Fourier transforms that

\[
Q \perp (T^n f)(\lambda), \quad n = 0, 1, 2, \ldots, \quad \text{where} \quad f(t) = m_c(t), \quad c \geq 0.
\]

Thus, by (11) and (13), \( Q \perp K_+^n e^{\bar{\iota}c} \int_0^\infty e^{-\bar{\iota}t\bar{k}(t)}dt \), for \( c \geq 0 \), that is,

\[
Q \perp K_+^{n+1} e^{\bar{\iota}c} - K_+^n e^{\bar{\iota}c} \int_0^\infty e^{-\bar{\iota}t\bar{k}(t)}dt \quad (n = 0, 1, 2, \ldots).
\]

Since \( Q \) and \( e^{\bar{\iota}c} \int_0^\infty e^{-\bar{\iota}t\bar{k}(t)}dt \) belong to \( R_- \) and \( R_+ \) respectively, it follows from (17) for \( n = 0 \) that \( Q \perp K_+ e^{\bar{\iota}c} \) (therefore \( Q \perp K_+ R_+ \)) and hence, by induction, that

\[
Q \perp e^{\bar{\iota}c} K_+^n, \quad n = 1, 2, \ldots, \quad c \geq 0.
\]

Relations (14) and (18) and the fact that \( Q \perp R_+ \) imply that \( Q \perp K^N(\lambda)R_+ \) for \( n = 0, 1, 2, \ldots \), that is,
(19) \[ \int_{-\infty}^{\infty} K_n(\lambda) F_+(\lambda) \overline{Q}(\lambda) d\lambda = 0 \quad \text{for } n = 0, 1, 2, \cdots, \]

where \( K(\lambda) \) is given in (6) and \( F_+(\lambda) \) is an arbitrary element of \( R_+ \).

Since \( k(t) \in L^1(-\infty, \infty) \), the function \( K(\lambda) \) is continuous and satisfies

\[ K(\lambda) \to 0 \quad \text{as} \quad |\lambda| \to \infty. \]

Also, by (2), \( K(\lambda) \) is real. Let \( f(K) \) denote the characteristic function of the \( K \)-set: \( |K| \geq 1/n \), where \( n \) is a positive integer. It follows from (19), Weierstrass’ approximation theorem, and the fact that \( F_+ \overline{Q} \) is in \( L^1(-\infty, \infty) \), that \( \int_{-\infty}^{\infty} f(K(\lambda)) F_+(\lambda) \overline{Q}(\lambda) d\lambda = 0 \) and hence

\[ \int_{E_n} F_+(\lambda) \overline{Q}(\lambda) d\lambda = 0, \]

where \( E_n = \{ \lambda : |K(\lambda)| \geq 1/n \} \). Since, for \( c \geq 0 \), \( e^{i\lambda c} \overline{Q}(\lambda) \) is in \( R_+ \), one can choose \( F_+ \) in (21) to be \( e^{i\lambda c} \overline{Q} \) and so \( \int_{E_n} e^{i\lambda c} \overline{Q}^2 d\lambda = 0 \). In view of (20),

\[ E_n \text{ is a bounded set.} \]

Since \( \overline{Q}^2 \) is in \( L^1(-\infty, \infty) \), one can therefore differentiate under the last integral with respect to \( c \) and let \( c \to 0^+ \) to obtain \( \int_{E_n} \lambda^m \overline{Q}^2 d\lambda = 0 \) \( (m = 0, 1, 2, \cdots ; n = 1, 2, \cdots) \). Again, using (22) and Weierstrass’ theorem, one concludes that \( Q(\lambda) = 0 \) a.e. on \( E_n \). In virtue of (6), the set \( \bigcup_{n=1}^{\infty} E_n \) differs from \( (-\infty, \infty) \) by a set of measure zero and hence \( Q(\lambda) = 0 \) a.e. on \( (-\infty, \infty) \). This implies that \( q(t) = 0 \) a.e. on \( 0 \leq t < \infty \), a contradiction. Thus \( \mathcal{M}_T = \mathcal{S} \) and so \( \mathcal{S}_a(A) = \mathcal{S} \) as was to be proved.

References


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