A THEOREM ON RANK WITH APPLICATIONS TO MAPPINGS ON SYMMETRY CLASSES OF TENSORS

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1. Results. Let $R$ be a field containing a real closed subfield $R_0$. The main results of this announcement follow.

**Theorem 1.** Let $A_1, A_2, \ldots, A_p$ be $m \times n$ matrices with entries in an infinite subset $\Omega$ of $R$ containing the natural numbers in $R_0$. Let $k$ be a positive integer and assume that the rank of each $A_i$ is at least $k$. Then there exist nonsingular matrices $E$ and $F$ with entries in $\mathbb{Q}$ such that every set of $k$ rows (columns) of $EA_iF$ is linearly independent, $i=1, \ldots, p$.

**Corollary 1.** If the matrices $A_1, \ldots, A_p$ in Theorem 1 each have rank precisely $k$ then every $k$-square subdeterminant of $EA_iF$ is nonzero, $i=1, \ldots, p$.

**Theorem 2.** If $A_1, \ldots, A_p$ are $n$-square complex Hermitian matrices all of rank at least $k$ then there exists a nonsingular matrix $E$ such that every set of $k$ rows (columns) of $E^*A_iE$ is linearly independent.

In 1933, J. Williamson [1] gave necessary and sufficient conditions for the compounds of two matrices to be equal. The nontrivial part of his result states the following: if $A$ is a complex matrix of rank $r$ and $r > m$ then $C_m(A) = C_m(B)$ if and only if $A = zB$ where $z^m = 1$. A result closely connected to Theorem 1 and generalizing the Williamson result can be proved. We state our theorem in an invariant setting.

Thus, let $V$ be an $n$-dimensional space over the complex numbers, let $H$ be a subgroup of the symmetric group $S_m$, $m \leq n$, and let $\chi$ be a complex valued character of degree 1 on $H$. A multilinear function $f(v_1, \ldots, v_m)$ is symmetric with respect to $H$ and $\chi$ if

\[ f(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \chi(\sigma)f(v_1, \ldots, v_m) \]

for all $v_1, \ldots, v_m$ in $V$ and all $\sigma \in H$. Let $P$ be a vector space and $f$ a fixed multilinear function symmetric with respect to $H$ and $\chi$, $f: V \times \cdots \times V \to P$, such that for any multilinear function $g, g: V \times \cdots \times V \to U$, also symmetric with respect to $H$ and $\chi$, there exists a linear $h: P \to U$ that makes the following diagram commutative:

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Then the pair $P, f$ is called a symmetry class of tensors associated with $H$ and $\chi$, e.g., $H = S_m$, $\chi = \text{sgn}$, $P = \Lambda^m V, f(v_1, \ldots, v_m) = v_1 \Lambda \cdots \Lambda v_m$, the usual $m$th Grassmann product. If $T$ is a linear transformation on $V$ then one defines a linear transformation $h$ via the diagram (1) with $U = P$, $g(v_1, \ldots, v_m) = f(Tv_1, \ldots, Tv_m)$. In this case $h$ is called the transformation induced by $T$ and will be denoted here by $K(T)$. If $P = \Lambda^m V$ then $K(T)$ is the $m$th compound of $T$, $C_m(T)$. Another example: if $H$ is the identity group then $P = \otimes_{i=1}^m V$, the $m$th tensor space over $V$, and $K(T) = \Pi^m(T)$, the $m$th Kronecker power of $T$.

We have the following generalization of Williamson’s result to an arbitrary symmetry class of tensors as described above. We do not present a proof here but this generalization depends directly on Theorem 1 for the case $p = 2$.

**Theorem 3.** If the rank of $T$ is $r$ and $r > m$, then $K(T) = K(S)$ if and only if $T = zS$ where $z^m = 1$.

**Corollary 2.** If $V$ is a unitary space, the rank of $T$ is $r$, and $r > m$, then $T$ is normal if and only if $K(T)$ is normal.

2. **Proof outline.** We say that a set of $m \times n$ matrices $(A_1, \ldots, A_p)$ have property $R_k$ if there exists a nonsingular $n$-square matrix $F$ such that every set of $k$ columns of $A_i F, i = 1, \ldots, p$, is linearly independent: this is abbreviated $(A_1, \ldots, A_p) \in R_k$. It is clear that if we can prove that any set of $p$ matrices all of rank at least $k$ satisfy $(A_1, \ldots, A_p) \in R_k$ then Theorem 1 will follow. Observe that if $S_1, \ldots, S_p$ are nonsingular $m$-square matrices then

\begin{equation}
(S_1 A_1, \ldots, S_p A_p) \in R_k
\end{equation}

if and only if $(A_1, \ldots, A_p) \in R_k$.

Now let $L$ be the $n$-square matrix whose $(i, j)$ entry is $i^j, i, j = 1, \ldots, n$. It is routine to verify that every subdeterminant of every order of $L$ is nonzero. Next, let $t_1, \ldots, t_n$ be independent indeterminates over $R$ and define an $n$-square matrix $L(t_1, \ldots, t_n)$ over $R[t_1, \ldots, t_n]$ whose $(i, j)$ entry is $t_i^j, i, j = 1, \ldots, n$. It follows that any specialization of $t_1, \ldots, t_n$ to nonzero elements of $\Omega$ pro-
duces a matrix every one of whose subdeterminants is nonzero. According to (2) we can take $(A_1, \cdots, A_p) = (H_1, \cdots, H_p)$ where $H_i$ is the Hermite normal form of $A_i, i = 1, \cdots, p$. Next, consider the matrices $B_i = H_iL(t_1, \cdots, t_n)$ and define the polynomial $p_i(t_1, \cdots, t_n)$ to be the product of all $C_{n,k}$ entries in the first row of the $k$th compound matrix of $B_i$, i.e., $C_k(B_i) = C_k(H_i)C_k(L(t_1, \cdots, t_n))$. The fact that $A_i$ and hence $H_i$ has rank at least $k$ implies that there exists a specialization of $p_i$ which is not zero. Hence the polynomial

$$P(t_1, \cdots, t_n) = \prod_{i=1}^{p} p_i(t_1, \cdots, t_n)$$

is not zero. It follows from a standard theorem on polynomials that there exist nonzero elements $\xi_1, \cdots, \xi_n$ in $\Omega$ for which $P(\xi_1, \cdots, \xi_n) \neq 0$. In other words, there exist nonzero $\xi_1, \cdots, \xi_n$ in $\Omega$ for which every entry in the first row of each of $C_k(H_i)(\xi_1, \cdots, \xi_n)$ is nonzero, $i = 1, \cdots, p$. This means that every set of $k$ columns of each of $H_iL(\xi_1, \cdots, \xi_n)$ is linearly independent and proves the result.

The rest of the results announced above follow from Theorem 1.

Reference


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