
THE UNIVERSITY OF WISCONSIN

THE $C^*$-ALGEBRA GENERATED BY AN ISOMETRY

BY L. A. COBURN

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1. Introduction. In this paper, I determine the structure of any $C^*$-algebra generated by an isometry. Using a theorem of Halmos [3], the problem is reduced to the study of $C^*$-algebras $\mathfrak{a}(A)$ generated by $A$ and $A^*$ where (i) $A$ is unitary, (ii) $A = S_\alpha$ with $S_\alpha$ the shift of multiplicity $\alpha$, and (iii) $A = W \oplus S_\alpha$ with $W$ unitary. In case (i), the resulting algebra is isometrically *-isomorphic to the algebra $C(\sigma(A))$ of all complex-valued continuous functions on the spectrum of $A$ and nothing more need be said. In cases (ii) and (iii), it turns out that $\mathfrak{a}(A)$ is isometrically *-isomorphic to $\mathfrak{a}(S_\alpha)$ so that $\mathfrak{a}(A)$ is independent of $W$ and $\alpha$. In each of these cases, there is a unique minimal closed two-sided ideal $\mathfrak{s}(A)$ such that $\mathfrak{a}(A)/\mathfrak{s}(A)$ is isometrically *-isomorphic to $C(T)$, where $T$ is the perimeter of the unit circle. The ideal $\mathfrak{s}(A)$ is determined spatially in the cases $A = S_\alpha$ and $A = W \oplus S_\alpha$.

We begin with the notation. For our purposes, all Hilbert spaces are complex and all ideals are closed and two-sided. If $\{e_n: n = 0, 1, 2, \ldots\}$ is an orthonormal basis for a Hilbert space $H$ then the shift $S = S_1$ is defined by $Se_n = e_{n+1}$. By a shift of multiplicity $\alpha$ is meant the $\alpha$-fold direct sum $S \oplus S \oplus \cdots \oplus S$ acting on $H \oplus H \oplus \cdots \oplus H$. The orthogonal projection onto the one-dimensional subspace of $H$ spanned by $e_n$ is denoted by $P_n$.

If $H$ (or $H_i$) is a Hilbert space then $\mathfrak{B}(H)$ (or $\mathfrak{B}(H_i)$) denotes the algebra of all bounded operators with the usual norm topology and $\mathfrak{K}$ (or $\mathfrak{K}_i$) denotes the ideal of all compact operators. The natural quotient map from $\mathfrak{B}(H)$ to $\mathfrak{B}(H)/\mathfrak{K}$ ($\mathfrak{B}(H_i)/\mathfrak{K}_i$) to $\mathfrak{B}(H_i)/\mathfrak{K}_i$ is given by

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If $A$ is an operator in $\mathfrak{B}(H)$, the $C^*$-algebra generated by $A$ will be named $\mathfrak{A}(A)$ or just $\mathfrak{A}$ when there is no possible doubt about $A$. An operator $A$ is called a Fredholm operator if $\pi(A)$ is invertible. The set of all Fredholm operators in $\mathfrak{B}(H)$ is denoted by $\mathfrak{F}$. It is known [1] that $A$ is in $\mathfrak{F}$ if and only if $A$ has closed range and finite-dimensional null and defect spaces.

2. The algebra $\mathfrak{A}(S)$. Our first object is to determine the ideals of $\mathfrak{A}(S)$. For vectors $y$ and $z$ in $H$, we define the operator $T_{y,z}$ by

$$T_{y,z}(x) = (x, y)z.$$ 

It is well known that the smallest closed subspace of $\mathfrak{B}(H)$ containing all $T_{y,z}$ is just $\mathfrak{K}$.

**Theorem 1.** The algebra $\mathfrak{A}(S)$ contains the full ideal of compact operators $\mathfrak{K}$ and $\mathfrak{K} \subset \mathfrak{S}$ for every nontrivial ideal $\mathfrak{S}$ in $\mathfrak{A}(S)$.

**Proof.** Since $1 - SS^* = P_0$ is in $\mathfrak{K}$, we see that $\mathfrak{A} \cap \mathfrak{K}$ is a nontrivial ideal in $\mathfrak{A}$. Now suppose that $\mathfrak{S}$ is any nontrivial ideal in $\mathfrak{A}$. If $A \neq 0$ is in $\mathfrak{S}$ then $A^*A$ is also in $\mathfrak{S}$. For some $N \geq 0$ we have $\|Ae_N\| \neq 0$. Since $S^mP_0S^m = P_m$, we see that $P_m$ is in $\mathfrak{A}$ for all $m \geq 0$. Hence $P_NA^*AP_N$ is in $\mathfrak{S}$. But

$$P_NA^*AP_Nx = (A^*AP_Nx, e_N)e_N = (x, P_NA^*Ae_N)e_N = \|Ae_N\|^2P_Nx;$$

so $P_N$ is in $\mathfrak{S}$ and thus $S^mP_NS^m = P_0$ is in $\mathfrak{S}$.

Now given any $\epsilon > 0$ and $y$ in $H$ there is a polynomial $p(x)$ so that $\|p(S)v_0 - y\| < \epsilon$. It follows that the operator $T_{y,v_0}$ has the property that $\|P_0[p(S)]^*-T_{y,v_0}\| < \epsilon$. Thus, $T_{y,v_0}$ is in $\mathfrak{S}$.

**Corollary 1.1.** The algebra $\mathfrak{A}(S)$ is dense in $\mathfrak{B}(H)$ with the strong topology.

**Proof.** $\mathfrak{K}$ is strongly dense in $\mathfrak{B}(H)$. $\square$

**Corollary 1.2.** The shift $S$ has no reducing subspaces except the trivial ones $(0)$ and $H$.

**Proof.** Otherwise, by Corollary 1.1 there would be a proper subspace invariant under all the operators in $\mathfrak{B}(H)$. $\square$
We can now complete the ideal theory for \( \mathfrak{a}(S) \).

**Theorem 2.** The algebra \( \mathfrak{a}(S)/\mathcal{K} \) is *-isomorphic and isometric to \( C(T) \).

**Proof.** Since \( S^*S - SS^* = P_0 \) is in \( \mathcal{K} \), it is apparent that \( \mathfrak{a}/\mathcal{K} \) is an abelian \( C^* \)-algebra. Hence \( \mathfrak{a}/\mathcal{K} \) is *-isomorphic and isometric to \( C(X) \) where \( X \) is the maximal ideal space of \( \mathfrak{a}/\mathcal{K} \). Now \( \mathfrak{a}/\mathcal{K} \) is generated by \( \pi(S) \) and \( \pi(S^*) \) so \( X \) is homeomorphic to the spectrum of \( \pi(S) \) in \( \mathfrak{a}/\mathcal{K} \). By a theorem in [2], the spectrum of \( \pi(S) \) in \( \mathfrak{a}/\mathcal{K} \) is the set \( \{ \lambda : S - \lambda \text{ is not in } \mathfrak{a} \} \) and an elementary computation shows that this set is just the perimeter of the unit circle \( T \).

Theorems 1 and 2 determine the structure of the ideals of \( \mathfrak{a}(S) \) since the ideal theory for \( C(T) \) is well known.

### 3. The Algebra \( \mathfrak{a}(W \oplus S) \)

The next part of the program is to determine the structure of \( \mathfrak{a}(W \oplus S) \) where \( W \) is a unitary operator on \( H_1 \) and \( S \) is the shift on \( H_2 \) with \( H_1 \oplus H_2 = H \). We require a Lemma which may be of some intrinsic interest.

**Lemma.** If \( A \oplus B \) is in \( \mathfrak{a}(W \oplus S) \) then \( \| A \| \leq \| \pi_2(B) \| \leq \| B \| \).

**Proof.** There is a sequence of "polynomials" in two noncommuting "indeterminates,"

\[
p_n(x, y) = \sum a_{i_1i_2 \ldots i_k} x_1^{i_1} y_1^{i_2} x_2^{i_3} \ldots y_k^{i_k},
\]

such that \( p_n(W, W^*) \rightarrow A \) and \( p_n(S, S^*) \rightarrow B \) in the operator norm topology. Thus

\[
p_n(\pi_2(S), \pi_2(S^*)) \rightarrow \pi_2(B)
\]

since \( \pi_2 \) is norm-decreasing. Now applying the Gelfand transform to the abelian \( C^* \)-algebra generated by \( \pi_2(S) \), we see that \( \sup_{\lambda \in T} \| p_n(\lambda, \lambda) \| \rightarrow \| \pi_2(B) \| \) since the spectrum of \( \pi_2(S) \) in \( \mathfrak{a}(S)/\mathcal{K}_2 \) is \( T \) and the Gelfand transform is an isometry. On the other hand, applying the Gelfand transform to the \( C^* \)-algebra generated by \( W \), we see that \( \sup_{\lambda \in \sigma(W)} \| p_n(\lambda, \lambda) \| \rightarrow \| A \| \). Since \( \sigma(W) \subset T \), the desired result follows.

**Theorem 3.** The algebra \( \mathfrak{a}(W \oplus S) \) is isometrically *-isomorphic to \( \mathfrak{a}(S) \) under the mapping \( W \oplus S \rightarrow S \).

**Proof.** The mapping \( W \oplus S \rightarrow S \) extends to the "polynomials" described in the Lemma. The extension is clearly a *-homorphism. If \( \rho(x, y) \) is such a "polynomial" then

\[
\| \rho(W, W^*) \oplus \rho(S, S^*) \| = \max(\| \rho(W, W^*) \|, \| \rho(S, S^*) \|).
\]
But by the Lemma, $\|p(W, W^*)\| \leq \|p(S, S^*)\|$ so
$$\|p(W, W^*) \oplus p(S, S^*)\| = \|p(S, S^*)\|.$$  
Hence, the mapping extends to an isometry from $\mathcal{A}(W \oplus S)$ onto $\mathcal{A}(S)$ which is also a *-isomorphism.

**Corollary 3.1.** The algebra $\mathcal{A}(W \oplus S)$ has a unique minimal nontrivial ideal, $\mathcal{J}(W \oplus S)$, and $\mathcal{A}(W \oplus S)/\mathcal{J}(W \oplus S) \cong C(T)$.

**Proof.** This follows from the properties of $\mathcal{A}(S)$ established in Theorems 1 and 2.

It is of some interest to determine the minimal ideal $\mathcal{J}(W \oplus S)$ spatially. This can be done in a manner similar to Theorem 1.

**Theorem 4.** The minimal nontrivial ideal $\mathcal{J}(W \oplus S)$ in $\mathcal{A}(W \oplus S)$ is $\mathcal{J}(W \oplus S) = 0 \oplus \mathcal{K}_2 = \mathcal{K} \cap \mathcal{A}(W \oplus S)$.

**Proof.** Since $(W^* \oplus S^*)(W \oplus S) - (W \oplus S)(W^* \oplus S^*) = 0 \oplus P_0,$ we see that $\mathcal{K} \cap \mathcal{A}$ is a nontrivial ideal in $\mathcal{A}$. Now suppose $\mathcal{J}$ is any nontrivial ideal. By the Lemma, if $C \oplus D$ is a nonzero element of $\mathcal{J}$ then $D \neq 0$. Hence, for some $e_n$ in the basis $\{e_n: n = 0, 1, 2, \ldots\}$ for $H_2$, we have $\|De_n\| \neq 0$. The argument that $0 \oplus \mathcal{K}_2 \subset \mathcal{J}$ now finishes as in the proof of Theorem 1. Further, if $C \oplus D$ is in $\mathcal{K} \cap \mathcal{A}$ then $C$ is in $\mathcal{K}_2$ and $D$ is in $\mathcal{K}_3$. It follows from the Lemma that $\|C\| = 0$ so that $0 \oplus \mathcal{K}_2 = \mathcal{K} \cap \mathcal{A}$. □

4. **The general case.** For the case $A$ an arbitrary isometry, the algebra $\mathcal{A}(A)$ can now be determined. Using a decomposition due to Halmos [3], any isometry $A$ on $H$ is either (i) unitary, (ii) unitarily equivalent to a shift $S_\alpha$ of multiplicity $\alpha$, or (iii) unitarily equivalent to a direct sum $W \oplus S_\alpha$ where $W$ is unitary. In the first case, $\mathcal{A}(A)$ is isometrically *-isomorphic to $C(\sigma(A))$. In case (ii), the mapping $S \mapsto S_\alpha$ induces an isometric *-isomorphism between $\mathcal{A}(A)$ and $\mathcal{A}(S)$ so the theory of §2 carries over to $\mathcal{A}(A)$. In case (iii), the mapping

$$W \oplus S \mapsto W \oplus S_\alpha$$

induces an isometric *-isomorphism between $\mathcal{A}(A)$ and $\mathcal{A}(W \oplus S)$ so the theory of §3 carries over to $\mathcal{A}(A)$. In cases (ii) and (iii), $\mathcal{A}(A) \cong \mathcal{A}(S)$ and there is a unique minimal ideal $\mathcal{J}(A) \neq 0$ with $\mathcal{A}(A)/\mathcal{J}(A) \cong C(T)$. Thus the algebraic structure is independent of $W$ and $\alpha$.

One can hope that knowing the ideals of $\mathcal{A}(A)$ makes possible a
classification of the \( \star \)-representations of \( \mathcal{A}(A) \). In fact, the representation theory for \( \mathcal{A}(S) \) can be handled by use of Theorem 1 and standard results on representations of \( \mathfrak{B}(H) \) and \( \mathfrak{K} \). In particular, using results from [4, p. 296] we see that every representation of \( \mathcal{A}(S) \) is a direct sum of identity representations and representations of \( C(T) \). Using the fact that for \( A \) an isometry, either \( \mathcal{A}(A) \cong C(\sigma(A)) \) or \( \mathcal{A}(A) \cong \mathcal{A}(S) \), the \( \star \)-representations for \( \mathcal{A}(A) \) can now be determined.

REFERENCES


Belfer Graduate School of Science, Yeshiva University