THE OBSTRUCTION TO FIBERING A MANIFOLD
OVER A CIRCLE

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1. Introduction. In [1], Stallings considers the following question. When does a 3-manifold fiber a circle? Browder and Levine generalized Stallings’ result to differentiable and piecewise linear manifolds $M$ of dimension greater than five under the restriction that $\pi_1(M) \cong \mathbb{Z}$. Their theorem is purely homotopic in nature. That is if $h: M' \to M$ is a homotopy equivalence and $f: M \to S^1$ is a smooth fiber map then there always exists a smooth fiber map $f': M' \to S^1$ such that $f'$ is homotopic to $f \circ h$.

This result is false if we drop their restriction on the fundamental group. In particular let $N$ be the cartesian product of a 3-dimensional lens space $L$ with fundamental group $\mathbb{Z}_{p^2}$ and the torus $T^n$ where $n \geq 5$. Let $M = N \times S^1$ and $f: M \to S^1$ denote projection onto the second factor. Then there exists a manifold $M'$ and a homotopy equivalence $h: M' \to M$ (in fact we may take $M'$ to be $\Sigma$-cobordant to $M$) such that a smooth fiber map $f': M' \to S^1$ homotopic to $f \circ h$ cannot exist. This example is based on recent deep results of Bass and Murthy [3] concerning the structure of the projective class group. In a joint paper with W. C. Hsiang [4] we use this example to construct an $h$-cobordism $(W, M, M')$ which is not homeomorphic to $M \times [0, 1]$.

In this paper we will state necessary and sufficient conditions, in terms of a new obstruction theory, for a manifold $M^n$ ($n \geq 6$) to fiber a circle. No restrictions will be placed on the fundamental group of $M$. We will always work in the differential category, but the corresponding theorem is also true in the piecewise-linear category.

2. Description of obstructions. Let $M^n$ be a closed connected smooth manifold with $n \geq 6$. Let $f: M \to S^1$ be a continuous map. (Recall that the homotopy class of $f$ is an element of $H^1(M, \mathbb{Z})$.) We will state three properties about $f$ which are necessary and sufficient to guarantee the existence of a smooth fiber map $\tilde{f}: M \to S^1$ homotopic to $f$. For convenience we restrict our attention to maps $f$ such that $f_\ast: \pi_1(M) \to \pi_1(S^1)$ is onto. (This is equivalent to considering only indi-

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visible elements in $H^1(M, Z)$.) This corresponds geometrically to considering fibrations with connected fiber. Let $G = \ker f_\sharp$ and $X$ denote the covering space of $M$ corresponding to $G$. If $\tilde{f}$ exists then it is clear that the fiber of $\tilde{f}$ is homotopically equivalent to $X$. But the fiber of $\tilde{f}$ would be a closed smooth manifold. In particular it would be a finite C.W. complex. Hence we obtain

**Condition 1.** $X$ is dominated by a finite C.W. complex.

Let $(N^{n-1}, v)$ be a framed submanifold of $M$ which represents $f$ under the Pontrjagin-Thom construction. Let $M_N$ denote the manifold obtained by "cutting" $M$ along $N$. Then $\partial M_N$ consists of two copies of $N$ which we label $N'$ and $N''$. (See Figure 1.)

![Figure 1](image)

$M_N$ is a cobordism from $N'$ to $N''$. The pair $(N, v)$ is called a splitting of $M$. Let $1 < s < n - 2$, $s$ an integer. When Condition 1 holds it is always possible to find a splitting $(N, v)$ such that $(M_N, N')$ has a handlebody decomposition with handles of only two dimension $s$ and $s+1$. The proof of this uses essentially the same arguments as in [2]. Note that the existence of a smooth fiber map is equivalent to the existence of a splitting $(N, v)$ such that $(M_N, N')$ is diffeomorphic to $N \times [0, 1]$. Conditions 2 and 3 will guarantee the existence of such a splitting. Condition 2 will hold if and only if there exists a splitting $(N, v)$ such that $(M_N, N', N'')$ is an $h$-cobordism. Condition 3 will hold if and only if some such $h$-cobordism is a product.

We proceed to formulate Condition 2. From the exact sequence

$$0 \rightarrow G \rightarrow \pi_1(M) \rightarrow H \rightarrow 0$$

we see that $\pi_1(M)$ is a semidirect product of $G$ and $Z$ with respect to an automorphism $\alpha$ of $G$. ($\alpha$ is only well defined up to an inner automorphism but this is all right for our purposes.) If Condition 1 is satisfied then we can define an element $c(f)$ in an abelian group $C(Z(G), \alpha)$. $c(f)$ has the following property: $c(f) = 0$ if and only if there exists a splitting $(N, v)$ such that $(M_N, N', N'')$ is an
$h$-cobordism. The proof of this fact is quite long and relies heavily on handle body theory.

**Condition 2.** $c(f) = 0$.

If Condition 2 is satisfied then $\tau(M_N, N') \in \text{Wh}(G)$ is defined. But it may be possible to have a second splitting $(N_1, v_1)$ such that $(M_{N_1}, N_1', N_1')$ is an $h$-cobordism and $\tau(M_{N_1}, N_1') \neq \tau(M_N, N')$. Let $\alpha_*$ denote the automorphism of $\text{Wh}(G)$ induced by $\alpha$ (see [5]). Let $\tau(f)$ be the image of $\tau(M_N, N')$ in the group $\text{Wh}(G)/\{x - \alpha_*(x) \mid x \in \text{Wh}(G)\}$ under the quotient homomorphism. We can show that $\tau(f)$ is well defined (i.e. $\tau(f)$ is independent of the splitting $(N, v)$). Also $\tau(f) = 0$ if and only if there exists a splitting $(N, v)$ such that $\tau(M_N, N') = 0$. The proof of this fact makes use of Stallings’ realizability theorem for $h$-cobordisms (see [6]). But the $s$-cobordism theorem of Barden-Mazur-Smale states that $M_N$ is diffeomorphic to $N \times [0, 1]$ if and only if $\tau(M_N, N') = 0$. Therefore

**Condition 3.** $\tau(f) = 0$.

Summarizing we have the following theorem.

**Theorem.** There exists a smooth fiber map $\tilde{f} : M \to S^1$ homotopic to $f$ if and only if

1. $X$ is dominated by a finite C.W. complex,
2. $c(f) = 0$,
3. $\tau(f) = 0$.

**Note.** There exists a version of this theorem for manifolds with boundary where the boundary already fibers a circle.

3. **Properties of $C(R, \alpha)$.** If $R$ is a ring with identity and $\alpha$ is an automorphism of $R$ then by a Grothendieck construction we can define an abelian group $C(R, \alpha)$. $\bar{K}_0(R)$ is a direct summand of $C(R, \alpha)$. Denote by $\bar{C}(R, \alpha)$ the complementary summand. Write $c(f) = \sigma(f) + \bar{c}(f)$ where $\sigma(f) \in \bar{K}_0(R)$ and $\bar{c}(f) \in \bar{C}(R, \alpha)$. Then $\sigma(f)$ is the Novikov-Siebenmann-Wall obstruction to $X$ splitting differentiably as a cartesian product $N \times R$.

$R$ is called regular if it is Noetherian and every finitely generated $R$ module has a resolution of finite length by projective $R$ modules. If $R$ is regular then $\bar{C}(R, \alpha) = 0$. But this is not the general situation since Bass and Murthy have shown that $\bar{C}(Z(G), \text{id}) \neq 0$ for certain finitely generated abelian groups $G$. A particular example is $G = Z \oplus Z \oplus Z_4$.

As an example of the fibering theorem consider the case where $G = Z^n$. Then $Z(G)$ is regular and hence $\bar{C}(Z(G), \alpha) = 0$. Also it is
known that \( \tilde{K}_0(Z(G)) = 0 \) and \( \text{Wh}(G) = 0 \) (see [5]). Therefore Conditions 2 and 3 become vacuous. Also observe that Condition 1 is only a homotopy theoretic condition. In particular if \( M \) and \( M' \) are homotopically equivalent manifolds such that \( r_1(M) \) is free abelian then \( M \) fibers a circle if and only if \( M' \) fibers a circle.

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