COMPACTIFICATION OF STRONGLY COUNTABLE DIMENSIONAL SPACES

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In this paper all spaces, including compactifications, are separable metrizable. Recall the following definitions. A space $X$ is strongly countable dimensional if $X$ is a countable union of closed finite-dimensional subsets. $X$ is a $G_δ$ space if $X$ is a $G_δ$-set in each space in which it is topologically embedded. A space $Y$ is a pseudo-polytope if $Y = Σ_1 ∪ Σ_2 ∪ \cdots$, where each $Σ_i$ is a simplex, $Σ_i \cap Σ_j$ is either empty or a face of both $Σ_i$ and $Σ_j$, and $\text{diam } Σ_i → 0$ as $i → ∞$. The term map always denotes a continuous function. Other notation is as in [3] and [8].

In [5] Lelek proved that every $G_δ$-space $X$ has a compactification $dX$ such that $dX \setminus X$ is a pseudo-polytope. He then raised the question of whether every strongly countable dimensional $G_δ$ space $X$ has a strongly countable dimensional compactification. This paper answers that question in the affirmative. We first state some preliminary propositions.

**Proposition 1.** Let $M ⊂ X$ with $\dim M ≤ n$, and let $\{U_i \mid i = 1, 2, \cdots\}$ be a sequence of sets open in $X$ and covering $M$. Then there is a sequence $\{V_i \mid i = 1, 2, \cdots\}$ of sets open in $X$ and covering $M$ such that $\text{ord} \{V_i \mid i = 1, 2, \cdots\} ≤ n + 1$ and such that $V_{k(n+1)+j} \subset U_{k+1}$ for $k = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, n + 1$.

**Proof.** The proof involves only a slight extension of the argument on page 54 of [2].

**Proposition 2.** Let $G$ be an open subset of a totally bounded space $Y$, and let $M_1, M_2, \cdots, M_r$ be relatively closed subsets of $G$ with $\dim M_i = m_i < ∞$ for $i = 1, 2, \cdots, r$. Let $ε > 0$. Then there is a collection $\{G_i \mid i = 1, 2, \cdots\}$ such that $G = \bigcup_{i=1}^{∞} G_i$ and

(i) Each $G_i$ is open in $Y$.

(ii) $\{G_i \mid i = 1, 2, \cdots\}$ is star-finite.

(iii) $\overline{G_i} \subset G$ for $i = 1, 2, \cdots$.

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(iv) $\text{diam } G_i < \epsilon$ for $i = 1, 2, \ldots$ and $\text{diam } G_i \to 0$ as $i \to \infty$.

(v) $\text{ord} \{ G_i \mid G_i \text{ meets } M_1 \cup M_2 \cup \cdots \cup M_k \} \leq m_1 + 1 + m_2 + 1 + \cdots + m_k + 1$ for $k = 1, 2, \ldots, r$.

**Proof.** We sketch the proof of this proposition. From page 114 of [4] we get a collection $\{ G_i \mid i = 1, 2, \ldots \}$ satisfying (i)-(iv). Open covers satisfying (i)-(iv) and (v) for $k = 1, 2, \ldots, r$ are now defined inductively. By Proposition 1 there is a sequence $\{ V_i \mid i = 1, 2, \ldots \}$ of open sets covering $M_i$ such that $\text{ord} \{ V_i \mid i = 1, 2, \ldots \} \leq m_1 + 1$ and $V_{k(n+i)+j} \subseteq G_{k+1}$ for $k = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, m_k + 1$. The collection $\{ V_i \mid i = 1, 2, \ldots \} \cup \{ G_i \mid G_i \text{ meets } M_i \} \mid i = 1, 2, \ldots \}$ then satisfies (i)-(iv) and (v) for $k = 1$.

Suppose $\{ G_i \mid i = 1, 2, \ldots \}$ covers $G$, satisfies (i)-(iv) and (v) for each $k = 1, 2, \ldots, n$. Let $C = M_1 \cup M_2 \cup \cdots \cup M_n$. By Proposition 1 there is a sequence $\{ V_i \mid i = 1, 2, \ldots \}$ of open sets covering $M_{n} \setminus C$ such that $\text{ord} \{ V_i \mid i = 1, 2, \ldots \} \leq m_n + 1$ and $V_{k(n+i)+j} \subseteq G_{k+1}$ for $k = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, m_{n+1} + 1$. The collection $\{ G_i \setminus (C \cup M_{n+1}) \mid i = 1, 2, \ldots \} \cup \{ V_i \mid i = 1, 2, \ldots \} \cup \{ G_i \mid G_i \text{ meets } C \}$ satisfies (i)-(iv) and (v) for each $k = 1, 2, \ldots, n + 1$. This completes inductive step and the sketch of the proof.

We are now in a position to prove the first theorem.

**Theorem 1.** Let $C$ be a closed subset of a compact space $Y$, and let $M_1, M_2, \ldots, M_r$ be closed subsets of $Y$ with $\text{dim } M_i = n_i < \infty$ for $i = 1, 2, \ldots, r$. Let $\epsilon > 0$. Then there is an $\epsilon$-map $f : Y \to \mathbb{R}^n$ such that $f(C) \cap f(Y \setminus C) = \emptyset$, $f \mid C$ is a homeomorphism, $f(Y \setminus C)$ is a countable polytope $P$, and $\text{dim } (M_i \setminus C) \leq m_i + 1 + m_{i+1} + \cdots + m_r$ for $i = 1, 2, \ldots, r$. Further, $P = \Sigma_1 \cup \Sigma_2 \cup \cdots$ where each $\Sigma_i$ is a simplex and $\text{diam } \Sigma_i \to 0$ as $i \to \infty$.

**Proof.** We may assume that $Y \subseteq \mathbb{R}^n$ and that the first coordinate of each point of $Y$ is zero. Let $\mathcal{G} = \{ G_i \mid i = 1, 2, \ldots \}$ be the open cover of $Y \setminus C$ given by Proposition 2 with $\text{diam } G_i < \epsilon/8$ for $i = 1, 2, \ldots$. For each $i$ such that $G_i \neq \emptyset$ pick a point $g_i \in G_i$. Then pick points $p_i$ with first coordinate greater than zero such that $d(p_i, g_i) < \min \{ 1/i, \epsilon/8 \}$ and such that $\{ p_i \mid i = 1, 2, \ldots \}$ is in general position. Let $N$ be the collection of simplices spanned by finite subsets $\{ p_{i_1}, p_{i_2}, \ldots, p_{i_m} \}$ where $G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_m} \neq \emptyset$. The points $\{ p_i \mid i = 1, 2, \ldots \}$ may be picked in such a way that $N$ is a CW-polytope, and certainly $N \cap Y = \emptyset$. Also, $N = \Sigma_1 \cup \Sigma_2 \cup \cdots$ where each $\Sigma_i$ is a simplex and $\text{diam } \Sigma_i \to 0$ as $i \to \infty$. Define $f' : Y \to \mathbb{R}^n$ by...
$f'(x) = \begin{cases} \sum_{i=1}^{\infty} d(x, Y \backslash G_i) p_i & \text{if } x \notin C, \\ \sum_{i=1}^{\infty} d(x, Y \backslash G_i) & \text{if } x \in C. \end{cases}$

It is not hard to show that $f'$ is continuous, and that $d(z, f'(z)) < \epsilon/4$ for each $z \in Y$. Triangulate $N$ into simplexes of diameter less than $\epsilon/4$. By a suitable induction, a map $f_i: f'(Y) \cap N \to N$ may be defined in such a way that $f'(y)$ and $f_if'(y)$ are in the same simplexes and $f_i(f'(Y) \cap N)$ is a subpolytope $P$ of $N$. The map $f: Y \to I^\omega$ defined by

$$f(z) = \begin{cases} z & z \in C, \\ f_if'(z) & z \in Y \backslash C \end{cases}$$

is then an $\epsilon$-map such that $f(C) \cap f(Y \backslash C) = \emptyset$, $f|C$ is a homeomorphism, and $f(Y \backslash C)$ is the desired polytope $P$. Finally, let $y \in M_i \backslash C$. By the conditions on the cover $\mathcal{G}$, $y$ is in at most $m_1 + 1 + m_2 + 1 + \cdots + m_i + 1$ elements of $\mathcal{G}$. Thus $f'(y)$, and hence also $f_if'(y)$, is in a simplex of dimension not greater than $m_1 + 1 + m_2 + 1 + \cdots + m_i$. Since $P$ is a countable polytope, $\dim f(M_i \backslash C) \leq m_1 + 1 + m_2 + 1 + \cdots + m_i$. Q.E.D.

Theorem 1 now enables us to prove our main theorem.

**Theorem 2.** Let $X$ be a strongly countable dimensional $G_\delta$ space. Then there is a strongly countable dimensional compactification $dX$ of $X$ such that $dX \backslash X$ is a pseudo-polytope.

**Proof.** Let $X = F_1 \cup F_2 \cup \cdots$ where $F_i$ is closed and dim $F_i = m_i < \infty$ for $i = 1, 2, \ldots$. By a result of Hurewicz [1] there is a compactification $cX$ of $X$ such that dim $\overline{F_i}^c = m_i$ for $i = 1, 2, \ldots$. Let $n_i = m_1 + 1 + m_2 + 1 + \cdots + m_i$. Since $X$ is a $G_\delta$ space, $cX \backslash X = Y_1 \cup Y_2 \cup \cdots$ where each $Y_i$ is compact and $Y_i \subseteq Y_{i+1}$ for $i = 1, 2, \ldots$. Let $Y_0 = \emptyset$. By Theorem 1 there is a $1/i$-map $f_i: Y_i \to I^\omega$ such that $f_i(Y_{i-1}) \cap f_i(Y_i \backslash Y_{i-1}) = \emptyset$, $f_i|Y_{i-1}$ is a homeomorphism, $f_i(Y_i \backslash Y_{i-1})$ is a countable polytope $P$, and dim $f_i(\overline{F_i}^c \cap (Y_i \backslash Y_{i-1})) \leq n_k$ for $k = 1, 2, \ldots, i$.

Decompose $cX$ into sets $f_i^{-1}(z)$ for $z \in f_i(Y_i \backslash Y_{i-1})$ and into individual points $x \in X$. Let the quotient space be $dX$ and let $f: cX \to dX$ be the quotient map. It may be shown that the decomposition of $cX$ is upper semicontinuous, so that $f$ is a closed map. Hence $dX$ is a
compactification of $X$. Furthermore, it is easily shown that there is a uniformly continuous homeomorphism $g_i: f_i(Y_i \setminus Y_{i-1}) \to f(Y_i \setminus Y_{i-1})$. Since $f_i(Y_i \setminus Y_{i-1})$ is a countable polytope for $i=1, 2, \ldots$, $dX \setminus X$ is a pseudo-polytope.

To show that $dX$ is strongly countable dimensional it is enough to show that $F_{dX}$ is strongly countable dimensional for $i=1, 2, \ldots$. Fix a positive integer $k$. Since $f$ is a closed map, $F_{dX} = f(F_kX) = F_k \cup \bigcup_{i=1}^n f(F_kX \cap (Y_i \setminus Y_{i-1}))$. Also, $f(F_kX \cap Y_{k-1}) \subset f(cX \setminus X) = dX \setminus X$, so $f(F_kX \cap Y_{k-1})$ is strongly countable dimensional. Let $C_n = \bigcup_{j=1}^n f(F_kX \cap (Y_j \setminus Y_{j-1}))$ for $n=k, k+1, \ldots$ and let $D_k = \bigcup_{j=1}^n C_j$. Each $C_j$ is closed in $D_k$. Further, $\dim C_k = \dim f(F_kX \cap (Y_k \setminus Y_{k-1})) \leq n_k$. Suppose $\dim C_i \leq n_k$. Then $C_{i+1} = C_i \cup f(F_kX \cap (Y_{i+1} \setminus Y_i))$, $C_k$ is closed in $C_{i+1}$, and $\dim f(F_kX \cap (Y_{i+1} \setminus Y_i)) \leq n_k$, so $\dim C_{i+1} \leq n_k$. Therefore $\dim D_k \leq n_k$, and $\dim D_k \cup F_k \leq n_k + m_k + 1$. $D_k \cup F_k$ is open in $F_{dX}$, so by Proposition 2 $D_k \cup F_k = \bigcup_{i=1}^{m_k} G_{ki}$, where $G_{ki} \subset D_k \cup F_k$ for $i=1, 2, \ldots$. Hence $\dim G_{ki} \leq n_k + m_k + 1$, and $F_{dX}$ is strongly countable dimensional. Q.E.D.

Sklyarenko gives an example in [9] which shows that being a $G_\delta$ space is a necessary hypothesis in Theorem 2.

BIBLIOGRAPHY


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