ON TAUBERIAN CONDITIONS OF TYPE $o$

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The series $\sum a_n$ ($\sum$ means $\sum_{n=0}^{\infty}$) is said to be summable to the sum $s$ by Abel's method of summability, if $\sum a_n x^n = f(x)$ is convergent for $0 < x < 1$ and if $f(x) \to s$ as $x \to 1^-$ ($x$ real). A classical theorem of A. Tauber [2] states that if $\sum a_n$ is summable to the sum $s$ by Abel's method and if

$$a_n = o(1/n) \quad \text{as} \quad n \to \infty$$

then $\sum a_n = s$. In today's language we put this in the following way: (1) is a Tauberian condition for Abel's method (cf., e.g., Hardy [1, pp. 149-152]). Again according to Tauber [2] the weaker condition

$$\delta_n = o(1) \quad \text{with} \quad \delta_n = (n + 1)^{-1} \sum_{k=0}^{n} k a_k$$

is also a Tauberian condition for Abel's method.

We shall show that Tauber's passage from (1) to (2) is possible for a very general class of summability methods. Formula (3) which yields this passage was already used by Tauber [2, p. 276, (6)]; here we exploit it more fully.

The summability method $V$ is said to be regular if $\sum a_n = s$ implies $V(\sum a_n) = s$. $V$ is called additive if $V(\sum b_n) = t$ implies $V(\sum (a_n + b_n)) = s + t$.

**Theorem.** If (1) is a Tauberian condition for the regular and additive method $V$ then also (2) is a Tauberian condition for $V$.

**Proof.** We assume that (1) is a Tauberian condition for $V$ and that we have under consideration a given series $\sum a_n$ which is summable $V$ to the sum $s$ and for which (2) is fulfilled. We have to show that $\sum a_n = s$. Putting $b_0 = a_0$ and $b_n = \delta_n / n$ ($n = 1, 2, \ldots$) the equation

$$a_0 + \cdots + a_n = (b_0 + \cdots + b_n) + \delta_n \quad (n = 0, 1, \ldots)$$

is easily proved by induction. Together with $V(\sum a_n) = s$ and $V(-\delta n) = 0$, (3) gives $V(\sum b_n) = s$. Since $b_n = o(1/n)$ we conclude that $\sum b_n = s$ and further, again from (3), that $\sum a_n = s$. 926
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If, a sequence $\lambda$ being given ($\lambda_{n-1} < \lambda_n \to \infty$ as $n \to \infty$), (1) is replaced by

(1a) $a_n = o(1/\lambda_n)$ as $n \to \infty$

and (2) by

(2a) $(n + 1)^{-1}(\lambda_0 a_0 + \cdots + \lambda_n a_n) = o(1),$

the theorem is still true provided that

$$n/\lambda_n = O(1) \quad \text{and} \quad n(\lambda_{n+1} - \lambda_n)/\lambda_{n+1} = O(1).$$

Herewith the cases

$$\lambda_n = n \log n, \quad \lambda_n = n \log n \log \log n, \cdots$$

are covered. The theorem fails to remain true if $n/\lambda_n \to \infty$. A paper investigating these questions and similar ones is under preparation.

REFERENCES


TECHNISCHE HOCHSCHULE STUTTGART, WEST GERMANY