PROOF. Let \( G = \sum \{ G_n | n \in J \} \) where \( G_n \) is solvable of radical class \( n \). Then \( G \in \mathcal{B} \) and has radical class \( \omega \). Let \( H = \prod \{ H_k | k \in J, H_k \cong G \} \). \( H \) has a subgroup satisfying the hypothesis of Theorem 3. Hence \( H \in \mathcal{B} \). Consequently, \( H \in \mathcal{B} \).

Classes of groups satisfying the conditions of Theorems 4 and 5 include the classes \( SN^*, SI^* \), subsolvable and polycyclic.

**BIBLIOGRAPHY**


**UNIVERSITY OF KANSAS**

**ALGEBRAIZATION OF ITERATED INTEGRATION ALONG PATHS**

BY KUO-TSAI CHEN

Communicated by Saunders Mac Lane June 12, 1967

If \( \Omega \) is the vector space of \( C^\infty \) 1-forms on a \( C^\infty \) manifold \( M \), then iterated integrals along a piecewise smooth path \( \alpha: [0, l] \rightarrow M \) can be inductively defined as below:

For \( r \geq 2 \) and \( w_1, w_2, \ldots, \in \Omega \),

\[
\int_{\alpha} w_1 \cdot \cdot \cdot w_r = \int_0^1 \left( \int_{\alpha_t} w_1 \cdot \cdot \cdot w_{r-1} \right) w_r (\alpha(t), \alpha'(t)) dt
\]

where \( \alpha_t = \alpha| [0, t] \). (See [3].)

This note is based on the following algebraic properties of the iterated integration:

(a) \( (\int_{\alpha} w_1 \cdot \cdot \cdot w_r) (\int_{\alpha} w_{r+1} \cdot \cdot \cdot w_{r+s}) = \sum \int_{\alpha} w_1 \cdot \cdot \cdot w_{r+s} \) summing over all \((r,s)\)-shuffles, i.e. those permutations \( \sigma \) of \( \{ 1, \ldots, r+s \} \) with \( \sigma^{-1}(1) < \cdots < \sigma^{-1}(r), \ \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s) \).

(b) If \( \phi = \alpha(0) \) and if \( f \) is any \( C^\infty \) function on \( M \), then

\[
\int_{\alpha} f w = \int_{\alpha} (df) w + f(\phi) \int_{\alpha} w.
\]

\(^1\text{The work has been partially supported by the National Science Foundation under Grant NSF-GP-5423.}\)
(c) If $\beta$ is a piecewise smooth path starting from the end point of $\alpha$, then
\[
\int_{\alpha} w_1 \cdots w_r = \int_{\beta} w_1 \cdots w_r + \int_{\alpha} w_1 \int_{\beta} w_2 \cdots w_r + \cdots + \int_{\alpha} w_1 \cdots w_r.
\]

The author wishes to thank Professor S. Mac Lane for valuable suggestions.

1. Let $K$ be a commutative unitary ring and $\Omega$ a $K$-module. Elements of $\Omega$ will be denoted by $w, w_1, w_2, \cdots$. Let $T(\Omega) = \oplus_{r \geq 0} T^r(\Omega)$ be the tensor $K$-algebra based on $\Omega$. For $u, v \in T(\Omega)$, we shall write $uv = u \otimes v$.

Define the shuffle multiplication $\circ$ of $T(\Omega)$ by $(w_1 \cdots w_r) \circ (w_{r+1} \cdots w_{r+s}) = \sum w_{r(1)} \cdots w_{s(r+s)}$ summing over all $(r, s)$-shuffles $\sigma$. Under the shuffle multiplication, $T(\Omega)$ becomes a commutative unitary $K$-algebra denoted by $Sh(\Omega)$. (See [6].) Moreover $Sh(\Omega)$ has a comultiplication $\Delta$ given by
\[
\Delta(w_1 \cdots w_r) = \sum_{0 \leq i \leq r} (w_1 \cdots w_i) \otimes (w_{i+1} \cdots w_r).
\]

Here we set $w_1 \cdots w_r = 1$ when $r = 0$. Let $\epsilon \in \text{Hom}_K(T(\Omega), K)$ be such that $\epsilon 1 = 1$ and $\epsilon T^r(\Omega) = \{0\}$ for $r \geq 1$. With the comultiplication $\Delta$ and the counit $\epsilon$, $Sh(\Omega)$ is a Hopf $K$-algebra which may be taken as a dualization of the tensor (Hopf) algebra with the diagonal map as comultiplication.

2. For any commutative unitary $K$-algebra $A$, it will be required that the canonical map $K \to A$ is injective. For any $A$-module $\Omega$, it will be required that $1w = w$. We say that $d \in \text{Hom}_K(A, \Omega)$ is a differentiation (of $A$) if $d(fg) = gdf + fdg$, $\forall f, g \in A$. If $A'$ is also a commutative unitary $K$-algebra, denote by $\text{Alg}(A, A')$ the set of morphisms $A \to A'$ of unitary $K$-algebras.

Denote by $\mathcal{D}$ the category of "pointed" differentiations of $K$-algebras: The objects of $\mathcal{D}$ are pairs $(d, p)$, where $d : A \to \Omega$ is a differentiation and $p \in \text{Alg}(A, K)$. If $(d', p')$ with $d' : A' \to \Omega'$ is also an object of $\mathcal{D}$, the set of morphisms $(d, p) \to (d', p')$ will be denoted by $\text{Diff}(d, p; d', p')$ which consists of the pairs $(\phi, \delta), \phi \in \text{Alg}(A, A'), \delta \in \text{Hom}_K(\Omega, \Omega')$ such that $\delta d = d' \delta, \delta(fw) = (\delta f)(\delta w), \forall f \in A, w \in \Omega$, and $p = p' \delta$. 
3. For any $K$-module $\Omega$, one may regard $\text{Sh}(\Omega) \otimes \Omega$ as an $\text{Sh}(\Omega)$-module. Define $\delta = \delta(\Omega): \text{Sh}(\Omega) \to \text{Sh}(\Omega) \otimes \Omega$ such that $\delta 1 = 0$ and $\delta(w_1 \cdots w_r) = (w_1 \cdots w_{r-1}) \otimes w_r, r \geq 1$. Then $\delta$ is a surjective differentiation, and $\text{Sh}(\Omega) = \ker \delta \oplus \ker \delta$. Write $\epsilon = \epsilon(\Omega)$. The pair $(\delta, \epsilon)$ can be characterized by the next theorem.

**Theorem 1.** Let $(d', p')$ with $d': A' \to \Omega'$ be an object of $\mathcal{D}$ such that $d'$ is surjective and $A' = \ker d' \oplus \ker p'$. Then, given any $\theta \in \text{Hom}_K(\Omega, \Omega')$, there exists a unique $(\tilde{\theta}, \hat{\theta}) \in \text{Diff}(\delta, \epsilon; d', p')$ such that $\theta = \tilde{\theta} + \hat{\theta}$, where $\nu: \Omega \to \text{Sh}(\Omega) \otimes \Omega$ is given by $\nu(w) = 1 \otimes w$.

4. An ideal $J$ of $A$ is said to be a $d$-ideal if $dJ = AdJ + J$. If $J$ is a $d$-ideal, then $d$ induces a differentiation $d_J: A/J \to \Omega/dJ$.

**Proposition.** Let $p \in \text{Alg}(A, K)$. If $I = I(d, p)$ is the $K$-submodule of $\text{Sh}(\Omega)$ generated by $u(fw)v - (u \circ df)v - (pf)uv, \forall u, v \in \text{Sh}(\Omega), w \in \Omega, f \in A$, then $I$ is the smallest $\delta$-ideal of $\text{Sh}(\Omega)$ that contains all $fw - (df)w - (pf)w$.

It follows that $\delta$ induces a surjective differentiation $\Delta = \Delta(d, p): \text{Sh}(\Omega) / I \to \text{Sh}(\Omega) \otimes \Omega / \delta I$. On the other hand, $\epsilon$ induces $E = E(d, p) \in \text{Alg}(\text{Sh}(\Omega) / I, K)$ such that $\text{Sh}(\Omega) / I = \ker \Delta \oplus \ker E$. The pair $(\Delta, E)$ can be characterized by the next theorem.

**Theorem 2.** Let $$(\tilde{\chi}, \hat{\chi}) = (\tilde{\chi}(d, p), \hat{\chi}(d, p)) \in \text{Diff}(d, p; \Delta, E)$$ be given by $\tilde{\chi}f = pf + df + I, \forall f \in A$, and $\hat{\chi}w = 1 \otimes w + \delta I$. If $(d', p')$ is as given in Theorem 1, then, for any $(\tilde{\theta}, \hat{\theta}) \in \text{Diff}(d', p'; d', p')$, there exists one unique $(\tilde{\Theta}, \hat{\Theta}) \in \text{Diff}(\Delta, E; d', p')$ such that $(\tilde{\theta}, \hat{\theta}) = (\tilde{\Theta}, \hat{\Theta})$.

5. **Definition.** A $d$-path from $p$ is an element $\alpha \in \text{Alg}(\text{Sh}(\Omega), K)$ such that $\alpha(I) = 0$. The end point of $\alpha$ is $q \in \text{Alg}(A, K)$ given by $qf = pf + \alpha(df), \forall f \in A$.

Recall that $\xi$ is the comultiplication of $\text{Sh}(\Omega)$. For $\alpha, \beta \in \text{Alg}(\text{Sh}(\Omega), K)$, define $\alpha \beta = \alpha \otimes \beta) \xi$. Then $\alpha \beta = \epsilon \alpha = \alpha$. It can be shown that $\text{Alg}(\text{Sh}(\Omega), K)$ is a group under the above multiplication.

**Theorem 3.** If $\alpha$ and $\beta$ are $d$-paths from $p$ to $q$ and from $q$ to $q'$ respectively, then $\alpha \beta$ is a $d$-path from $p$ to $q'$; and $\alpha^{-1}$ is a $d$-path from $q$ to $p$.

6. We say that $A$ is $d$-connected if, for any $p, q \in \text{Alg}(A, K)$, there exists a $d$-path from $p$ to $q$.

**Proposition.** If $A$ is $d$-connected and if $p, q \in \text{Alg}(A, K)$, then $(\Delta(d, p), E(d, p)) \cong (\Delta(d, q), E(d, q))$ in the category $\mathcal{D}$. 
PROPOSITION. If $\text{Alg}(A, K)$ and $\text{Alg}(A', K)$ are both nonempty, then $A \oplus A'$ is not $(d \oplus d')$-connected.

There is a partial converse to the above assertion which states that if $\text{Alg}(A, K)$ is the disjoint union of two nonempty sets such that there exists no $d$-path with its initial point in one of the sets and its end point in the other, then, under reasonable conditions, $A$ is non-trivially imbedded in a direct sum.

PROPOSITION. If $A$ is $d$-connected with nonempty $\text{Alg}(A, K)$ and if $d$ is surjective, then $A$ is a $d$-tree, i.e. $A$ has no closed $d$-path other than $e$.

BIBLIOGRAPHY


STATE UNIVERSITY OF NEW YORK AT BUFFALO AND UNIVERSITY OF ILLINOIS