CHARACTERIZATIONS OF THE ESSENTIAL SPECTRUM OF F. E. BROWDER

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Let $T$ be a densely defined closed linear operator on a Banach space $X$. F. E. Browder [1] has defined the essential spectrum of $T$, $\text{ess}(T)$, to be the set of complex numbers $\lambda$ such that at least one of the following conditions is satisfied:

(i) The range $\mathcal{R}(\lambda - T)$ of the operator $\lambda - T$ is not closed in $X$;

(ii) $\bigcup_{k \geq 0} \mathcal{N}[(\lambda - T)^k]$ is of infinite dimension, ($\mathcal{N}(S)$ being the null space of the operator $S$);

(iii) The point $\lambda$ is a limit point of the spectrum of $T$.

In [7], M. Schechter discusses two other sets of complex numbers, $\sigma_{ev}(T)$ and $\sigma_{em}(T)$, which have also been called the essential spectrum of $T$ (cf. [10]). He characterizes $\sigma_{em}(T)$ as the largest subset of the spectrum of $T$ which remains invariant under compact perturbations of $T$. Although $\sigma_{em}(T)$ is in general a proper subset of $\text{ess}(T)$, Schechter gives conditions which guarantee that $\text{ess}(T)$ will remain invariant under compact (and certain other) perturbations of $T$. The proofs of these results usually reduce to showing that $\sigma_{em}(T) = \text{ess}(T)$.

In this paper we replace Schechter's conditions on $T$ by a condition on the perturbing operator and show that $\text{ess}(T)$ is invariant under compact (and certain other) perturbations of $T$, provided the perturbing operators commute with $T$. We shall say that a linear operator $C$ commutes with $T$ if (i) the domain of $C$, $\mathcal{D}(C)$, contains the domain of $T$, (ii) $Cx \in \mathcal{D}(T)$ whenever $x \in \mathcal{D}(T)$, (iii) and $TCx = CTx$ for $x \in \mathcal{D}(T)$. 

Following the notation and terminology of [9], we denote the dimension of the null space or nullity of an operator $S$ by $\nu(S)$ and the codimension of the range or defect of $S$ by $d(S)$. The ascent of $S$, $\alpha(S)$, is the smallest integer $p$ such that $\mathcal{N}(S^p) = \mathcal{N}(S^{p+1})$, and the descent of $S$, $\delta(S)$, is the smallest integer $q$ such that $\mathcal{R}(S^q) = \mathcal{R}(S^{q+1})$. (It may happen that $\alpha(S) = \infty$ or $\delta(S) = \infty$.) Suppose that $\lambda_0$ is a pole of order $p$ of the resolvent operator $(\lambda - T)^{-1}$ and let $E$ be the spectral projection corresponding to the spectral set $\{\lambda_0\}$. The range of $E$ is the null space of $(\lambda_0 - T)^p$ and the dimension of this space is called the rank of the pole $\lambda_0$.

**Theorem 1.** Let $T$ be a densely defined closed linear operator on a
Banach space $X$. Let $\lambda_0$ be a point of the spectrum of $T$. The following statements are equivalent:

1. $\lambda_0$ is not in $\text{ess}(T)$.
2. $\lambda_0$ is a pole of the resolvent $(\lambda - T)^{-1}$ of finite rank.
3. $\lambda_0$ has finite ascent, descent, and defect.
4. $n(\lambda_0 - T) = a(\lambda_0 - T) < \infty$ and $a(\lambda_0 - T) < \infty$.

The equivalence of (1) and (2) was proved by F. E. Browder in Lemma 17 of [1]. Since that time results of Kaashoek [3], Taylor [9] and the author [6] have provided tools for giving a short proof of the equivalence of (1), (2), and (4) without requiring that $T$ have a dense domain.

**Theorem 2.** Let $T$ be a closed linear operator on a Banach space $X$. A point $\lambda_0$ in the spectrum of $T$ is a pole of the resolvent of finite rank if and only if there is a compact linear operator $C$ with $C(\sigma(T)) \subset \sigma(T)$ and $TCx = CTx$ for $x \in \sigma(T^2)$ such that $\lambda_0 - (T + C)$ has a bounded inverse defined on all of $X$.

**Corollary.** Let $T$ be a closed linear operator on a Banach space $X$. Then $\text{ess}(T)$ is the largest subset of the spectrum $\sigma(T)$ which remains invariant under perturbations of $T$ by compact operators which commute with $T$, i.e.

$$\text{ess}(T) = \{ \lambda \mid \lambda \in \sigma(T + C) \text{ for every compact operator } C \text{ such that } C(\sigma(T)) \subset \sigma(T) \text{ and } TCx = CTx \text{ for } x \in \sigma(T^2) \}. $$

Both $\sigma_{\text{em}}(T)$ and $\text{ess}(T)$ are also invariant under certain unbounded perturbations. Suppose that $T$ is a closed linear operator in $X$ and $C$ is a linear operator with $\sigma(C) \supset \sigma(T)$. We say that $C$ is $T$-closable if $x_n \to 0$, $Tx_n \to 0$, $Cx_n \to s$ for $\{x_n\} \subset \sigma(T)$ implies $s = 0$. The operator $C$ is $T$-compact if for any sequence $\{x_n\} \subset \sigma(T)$ satisfying

$$\|x_n\| + \|Tx_n\| \leq \text{const.},$$

the sequence $\{Cx_n\}$ has a convergent subsequence. The operator $C$ is $T$-pseudo-compact if for any sequence $\{x_n\} \subset \sigma(T)$ satisfying

$$\|x_n\| + \|Tx_n\| + \|Cx_n\| \leq \text{const.},$$

the sequence $\{Cx_n\}$ has a convergent subsequence. Theorem 3 below is the analogue of Theorems 2.1 and 2.2 of [7] (although in that paper $T$ and $C$ were densely defined).

**Theorem 3.** Let $T$ be a closed linear operator on a Banach space $X$. Then $\text{ess}(T)$ is the largest subset of the spectrum of $T$ which remains in-
variant under perturbations of $T$ by operators $C$ which commute with $T$ and are either $T$-compact or are $T$-closable and $T$-pseudo-compact.

The compactness condition on the perturbing operator $C$ may be generalized in another direction. It is well known that

$$n(T) - d(T) = n(T + C) - d(T + C)$$

when $C$ is a strictly singular operator (or is strictly singular relative to $T$, cf. [4]), or $C$ is an inessential operator (cf. [5]). From Schechter's characterization [7] of $\sigma_{em}(T)$ as the complement in the complex plane of the set of points $\lambda$ for which $n(\lambda - T) = d(\lambda - T) < \infty$ it follows immediately that $\sigma_{em}(T)$ is the largest subset of the spectrum which remains invariant under perturbations of $T$ by strictly singular or inessential operators. Even more is true in the analogous situation for $\text{ess}(T)$. The ideal of strictly singular operators and the ideal of inessential operators are both contained in a set of bounded linear operator called Riesz operators [2].

A Riesz operator $R$ is characterized by the property that it is a bounded linear operator with $d(\lambda - R) < \infty$ for all $\lambda \neq 0$ [6].

**Theorem 4.** Let $T$ be a closed linear operator on a Banach space $X$. Then $\text{ess}(T)$ is the largest subset of the spectrum of $T$ which remains invariant under perturbations of $T$ by Riesz operators $R$ which commute with $T$.

Proofs of these results will appear elsewhere.

**References**