REPRESENTATION OF F-RINGS

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Consider a lattice ordered algebra $A$ with identity over the rationals $\mathbb{Q}$; $A$ is called an $f$-ring if $a \land b = 0$, $c \geq 0$, implies that $ca \land b = ac \land b = 0$. The maximal $l$-ideals $\mathcal{M}$ of $A$ form a compact Hausdorff space in the hull-kernel topology. If $A$ is archimedean, i.e. a so called $\Phi$-algebra, then it is known [5] that $A$ is isomorphic to a subalgebra of the partial algebra $D(\mathcal{M})$ of all continuous functions $f$: $\mathcal{M} \to \mathbb{R} \cup \{\pm \infty\}$ which are finite on a dense open set. The fact that there is a sizable theory of $\Phi$-algebras [5], [6], [10] with no counter part for the more general class of $f$-rings may partly be due to the existence of this representation, $A \subseteq D(\mathcal{M})$ for $\Phi$-algebras, and a lack of such a representation in the nonarchimedean case. This latter representation has the defect that it is not onto. Even when $D(\mathcal{M})$ is an algebra, $A$ need not be all of $D(\mathcal{M})$. Our objective is to give a representation which not only corrects this defect, but also is applicable to a wider class of $f$-rings. This new representation will show that the "$f$" in the term "$f$-ring" is well justified.

Define $E = \bigcup \{A/M | M \in \mathcal{M}\}$; $\pi: E \to \mathcal{M}$, $\pi^{-1}(M) = A/M$. Each $a \in A$ gives a map $\delta: \mathcal{M} \to E$, $\delta(M) = a + M$. For any subset $A_1 \subseteq A$, set $\hat{A}_1 = \{\delta | a \in A_1\}$. In order that $A \cong \hat{A}$, the condition (A) will be assumed throughout to hold

(A) $\cap \mathcal{M} = \{0\}$.

Appropriate topologies can be introduced in $E$ and $\mathcal{M}$ making $\pi$ into a structure which generalizes sheaves and fiber bundles—a so called field. (For a complete theory of fields, see [3].) The topologies on $E$ and $\mathcal{M}$ are unique in a certain well-defined sense. Let $\Gamma(\mathcal{M}, E)$ be the $l$-group of all continuous cross sections $\sigma: \mathcal{M} \to E$ with $\pi \circ \sigma$ the identity on $\mathcal{M}$. Then $\pi$ is continuous and $\hat{A} \subseteq \Gamma(\mathcal{M}, E)$ is an $l$-subgroup. Let $A^*$ be the subalgebra $A^* = \{a \in A | |a| < r_1, \text{some } 0 < r \in \mathbb{Q}\}$. Then $\Gamma(\mathcal{M}, E)^* = \{\sigma \in \Gamma(\mathcal{M}, E) | |\sigma| < \delta \text{ for some } a \in A^*\}$ is a convex $l$-subgroup of $\Gamma(\mathcal{M}, E)$.

Although for ease of exposition, $A$ here is the additive group of a ring, the multiplicative structure of $A$ has not been used thus far. The above construction will be carried out more generally for an arbitrary $l$-group $A$ and any set of prime subgroups $\mathcal{M}$ with $\cap \mathcal{M} = \{0\}$.

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If $M$ is not normal in $A$, then $A/M$ is not a group but merely a right coset space.

Returning now to our previous assumptions, the algebra $A$ is an additive topological group with $\{a \in A \mid |a| < r\}$, $0 < r \in \mathbb{Q}$, as zero neighborhoods. It is possibly non-Hausdorff. If $A$ is complete in this uniform structure, it will be said to be uniformly closed. Under obvious pointwise operations, $\Gamma(M, E)$ is an $\ell$-group; $\hat{A}$ is said to be uniformly dense in $\Gamma(M, E)$ if for any $\sigma \in \Gamma(M, E)$ and any $0 < r \in \mathbb{Q}$, there is an $a \in A$ with $|a - \sigma| < r 1$. Since we are interested in cases when $\hat{A} = \Gamma(M, E)$, or when at least $\hat{A}$ is uniformly dense in $\Gamma(M, E)$, besides the assumption (A) various of the following hypotheses will have to be imposed:

(B) $A^*$ is closed under bounded inversion, i.e. if $1 < a \in A^*$, then $1/a \in A$.
(B') $A$ is closed under bounded inversion.
(C) $A$ is uniformly closed.

Since $A^* \subseteq C(M)$, the ring of real continuous functions, (A) and (C) imply (B).

For representation purposes it is important that the algebra has the property described in the next definition.

1. DEFINITION. A subset $A_1$ of $A$ contains positive bounded partitions of identity on $M$, if for any open cover $M = U_1 \cup \cdots \cup U_n$, there are $e_j \in A_1$ satisfying $e_j \subseteq \cap M \setminus U_j$; $0 \leq e_j \leq 1$ for $j = 1, \cdots, n$; and $1 = e_1 + \cdots + e_n$.

The proof of the next lemma is obtained by using the hull-kernel topology together with the lattice properties of $A$.

2. LEMMA. If conditions (A) and (B) hold, then $A^*$ contains positive bounded partitions of identity on $M$.

It should be noted that $A$ may be noncommutative even if $A^*$ is abelian.

3. LEMMA. If (A) and (B) hold, then the following conditions are all equivalent:
(i) $A^*$ is archimedean;
(ii) $A$ is Hausdorff;
(iii) $E$ is Hausdorff.

The next proposition is not only needed to identify the fibers, but it is also of independent interest.

4. PROPOSITION. Consider any totally ordered f-ring $A$ such that every $1 < a \in A$ has a two sided inverse in $A$. Let $I$ be the set of invertible ele-
ments and define $N$ as the set $N = \{ x \in A \mid |x| < i \text{ all } i \in I \}$. Then:

(i) $N$ is a maximal ideal of $A$ which is an $l$-ideal.
(ii) $A/N$ is a totally ordered division ring.

Very easily describable necessary and sufficient conditions for embedding a rational $f$-algebra into a real $f$-algebra do not seem to be available (see [7, p. 351, 2.9] and [11]).

5. Corollary. If the f-ring $A$ satisfies conditions (A) and (B'), then each $A/M$ is a totally ordered division ring. Furthermore, $A$ can be embedded in an $f$-algebra over the reals.

The previous lemmas are now used to obtain the main theorem.

6. Theorem. Suppose $A$ is an $f$-algebra with identity over the rationals $\mathbb{Q}$. Define $A^\ast = \{ a \in A \mid |a| < r \text{ for some } r \in \mathbb{Q} \}$ and $\mathcal{M}$ as the set of all maximal $l$-ideals of $A$. Assume that

(A) $\cap \mathcal{M} = \{ 0 \}$;
(B) $1 < a \in A^\ast \Rightarrow 1/a \in A^\ast$.

Let $\pi: E = \bigcup \{ A/M \mid M \in \mathcal{M} \} \rightarrow \mathcal{M}, \hat{A}^\ast, \hat{A}, \Gamma(\mathcal{M}, E)$, and $\Gamma(\mathcal{M}, E)^\ast$ be as in the introduction.

(i) There is a field $\pi$ where $\mathcal{M}$ has the hull-kernel topology. Each $A/M, M \in \mathcal{M}$, is a totally ordered integral domain. There are $l$-isomorphisms

$$A \rightarrow A \subseteq \Gamma(\mathcal{M}, E), \quad A^\ast \rightarrow A^\ast \subseteq \Gamma(\mathcal{M}, E)^\ast.$$ 

(ii) $\hat{A}$ is uniformly dense in $\Gamma(\mathcal{M}, E)$.

Now assume conditions (A) and (C), where

(C) $A$ is uniformly closed.

Then the following two assertions are valid:

(iii) $A^\ast \cong \hat{A}^\ast = C(\mathcal{M}) \hat{1} = \Gamma(\mathcal{M}, E)^\ast; E$ is Hausdorff.
(iv) $\hat{A} = \Gamma(\mathcal{M}, E)$.

By using Proposition 4 and imposing more hypotheses, we can obtain additional information in the above Theorem.

7. Corollary. With the same notation as in the previous theorem, assume (A) and (B'):

(B') $1 < a \in A \Rightarrow 1/a \in A$.

Then conclusions (i) and (ii) of the previous theorem hold. Furthermore, each $\pi^{-1}(M), M \in \mathcal{M}$, is a totally ordered division ring.

A converse theorem can also be formulated. One starts from a field $\pi: E \rightarrow \mathcal{M}$ over a compact Hausdorff space whose stalks are totally ordered integral domains. Then an appropriate subalgebra $\Lambda$, in
$\Delta \subseteq \Gamma(\mathcal{M}, E)$ is shown to be an $f$-algebra satisfying the algebraic hypotheses (A) and (B) of the previous theorem.

The full proofs of these results will appear elsewhere later.

REFERENCES


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