CONSTRUCTIONS ON LOW-DIMENSIONAL
DIFFERENTIABLE MANIFOLDS

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1. This note contains the statements of three theorems on low-dimensional differentiable manifolds (dimensions 3 and 4). The proofs, which use techniques partly connected to [1], will appear elsewhere.

We denote by \( M_n + (\phi) \) the manifold obtained by adding the handle of index \( \lambda \), \( D_\lambda \times D_{n-\lambda} \), via the embedding \( \phi: S_{n-1, \lambda} \times D_{n-\lambda} \to \partial M_n \), to \( M_n \). More generally, we shall use the following notation: If \( P_{n-1} \subseteq S_{n-1} = \partial D_n \) is a bounded submanifold and \( \psi: P_{n-1} \to M_n \) an embedding, we denote by \( M_n + (\psi) \), the space \( M_n \cup D_n \), where every \( x \in P_{n-1} \) is identified to \( \psi(x) \in M_n \). It is understood that, if \( \psi(P_{n-1}) \subseteq \partial M_n \), then \( M_n + (\psi) \) is a “usual” differentiable manifold, otherwise a “singular” one (see §2).

THEOREM 1. Let \( M_3 \) be a compact, differentiable, homotopy 3-disk. Then \( M_3 \times I \) is diffeomorphic to \( D_4 \) with handles of index 2 and 3 added:

\[
M_3 \times I = D_4 + (\phi_2) + \cdots + (\phi_3) + (\phi_1) + \cdots + (\phi_5).
\]

Hence, one can eliminate the handles of index 1 of \( M_3 \times I \) (compare with the similar procedure, in higher dimensions [2]).

In fact we obtain Theorem 1 from the slightly stronger:

THEOREM 1'. If \( M_3 \) is a compact, differentiable homotopy 3-disk, there exists an integer \( p = p(M_3) \) such that:

\[
(M_3 \# (S_2 \times I) \# \cdots \# (S_2 \times I)(p \text{ times})) \times I = D_4 + (\phi_2) + \cdots + (\phi_5).
\]

This together with some immersion theory, implies easily the main result from [1].

The next theorem is the main step in proving Theorem 1'. But in order to state it, we need some preparation.

2. We consider the 3-manifold \( T_p = (S_1 \times D_2) \# (S_1 \times D_2) \# \cdots \# (S_1 \times D_2) \) \( (p \text{ times}) \) and its double \( 2T_p = (S_1 \times S_2) \# (S_1 \times S_2) \# \cdots \# (S_1 \times S_2) \) \( (p \text{ times}) \). \( \# \) means connected sum.) A family of \( p \) 2-by-2 disjoint embeddings \( \phi_i: S_1 \to T_p \) \( (i = 1, \cdots, p) \) is called

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"unknotted," if, after a diffeomorphism of $T_p$, the images become $(S_1 \times o) \cup (S_1 \times o) \cup \cdots \cup (S_1 \times o)$ ($\rho$ times, $o$ = center of $D_2$). Similarly, if $o \in S_2$, is some fixed point, a family of $p$, 2-by-2 disjoint embeddings $\psi_i: S_1 \rightarrow 2T_p$ $(i=1, \cdots, p)$ is called "unknotted" if after a diffeomorphism of $2T_p$, the images become $(S_1 \times o) \cup \cdots \cup (S_1 \times o)$ ($\phi$ times). (The difference between the two notions is illustrated by the Figures 1a, 1b, where "knotted" embeddings $S_1 \rightarrow T_1$ are presented, such that the composite embeddings $S_1 \rightarrow T_1 \subset 2T_1$ are unknotted. Figure 1b contradicts, unfortunately, the obvious conjecture suggested by Figure 1a.)

![Figure 1a](image1.png)  ![Figure 1b](image2.png)

We consider now (compact) 3-manifolds with singularities. These will be compact spaces $V_3$ which are everywhere (bounded) differentiable manifolds, except for a finite number of compact neighborhoods $W$, which admit descriptions of the following type: We consider two embeddings $\phi, \psi: I \rightarrow S_2 = \partial D_2$, such that $\phi(I) \cap \psi(I)$ consists of exactly one point, with transversal intersection, and two thin tubular neighborhoods around them: $\Phi, \Psi: I \times I \rightarrow S_2 = \partial D_2$. $I \times I$ is assimilated to $I \times I \times o \subset \partial (I \times I \times I) = \partial D_3$ and then $W$ is our original $D_3$ (target of $\phi, \psi$) with two other copies of $D_3$ added along $\Phi, \Psi$:

$$W = D_3 + (\Phi) + (\Psi) = D_3 + (\Psi) + (\Phi).$$

$V_3$ is "regular" except for a "singular" set $\sigma(V_3)$ which is a bounded 2-manifold, having as connected components various copies of $D_2$.

We consider resolutions (of singularities) for $V_3$, $\Pi: V'_3 \rightarrow V_3$ where $V'_3$ is a nonsingular 3-manifold, $\Pi^{-1}(x)$ has exactly 2 elements if $x \in \text{int } \sigma(V_3)$ and exactly 1 element if $x$ is regular. (If $x \in \partial \sigma(V_3)$, as we shall see in a moment, $\Pi^{-1}(x)$ has one point in half the cases and two in the other half.) It is moreover understood that, if $W$ is
as before, \( \Pi^{-1}(W) = W' \) is obtained by cutting the \( I \times I \times I \) corresponding to \( \Phi \), from \( D_3 + (\Psi) \) along \( \Phi^{-1}(\text{Image } \Phi \cap \text{Image } \Psi) \) (or the \( I \times I \times I \) corresponding to \( \Psi \), from \( D_3 + (\Phi) \)). So \( W' = S_1 \times D_3 \), and passing from \( W \) to \( W' \), Image \( \Phi \cap \text{Image } \Psi = I \times I \) "blows up" into \( \beta = \Phi^{-1}(\text{Image } \Phi \cap \text{Image } \Psi) + \Phi^{-1}(\text{Image } \Phi \cap \text{Image } \Psi) \), diffeomorphic to \( S_1 \times I \) (the first summand is in \( I \times I \times I \), the other in \( D_3 + (\Psi) \)). We say that \( \Phi \) (or \( \Psi \)) is specified in the resolution \( \Pi = V'_3 \to V_3 \). (One remarks that the two \( I \times I \times I = D_3 \) play a symmetric role in \( W \), but cannot be interchanged with the original \( D_3 \); this is easily seen by looking at the sheaf of local homology groups along \( \sigma(W) = \text{Image } \Phi \cap \text{Image } \Psi \).)

If \( \Phi \) is specified in the resolution \( \Pi : V'_3 \to V_3 \), as above, there exists a canonical embedding \( j : W' \to S_3 \) which is uniquely determined (up to isotopy) by the requirements that \( j(W') \) be unknotted and \( j(\beta) \) be contained in a nonsingular 2-disk of \( S_3 \).

Let us consider the category \( \mathcal{R} \) of resolutions \( \Pi : V'_3 \to V_3 \) (for all \( V'_3 \)'s) where the morphisms are given by commutative squares, having \( \Pi \) on the verticals and embeddings on the horizontals. Let us also consider the category \( \mathcal{C} \) consisting of triples \((M_4, j, M_3)\) where \( M_4 \) is a bounded differentiable 4-manifold, \( M_3 \) a (bounded) differentiable 3-manifold and \( j : M_3 \to \partial M_4 \) an embedding. Morphisms are again commutative squares having the \( j \)'s on the verticals and embeddings on the horizontals. We have:

**Lemma.** There exists a unique ("thickening") functor \( \Theta : \mathcal{R} \to \mathcal{C} \) such that, if \( \eta \in \mathcal{R} \) is \( \Pi : V'_3 \to V_3 \) then \( \Theta(\eta) = (\Theta_4(\eta), J(\eta), V'_3) \) and the following requirements are fulfilled:

(a) If \( V_3 \) is nonsingular (\( \sigma(V_3) = \emptyset \)) and \( \eta \) is the (only possible) resolution: identity: \( V'_3 \to V_3 \), then \( \Theta_4(\eta) = V_3 \times I \) and \( J(\eta) = V_3 \times o \cap \partial (V_3 \times I) = V_3 \times o + \partial V_3 \times I + V_3 \times 1 \).

(b) \( \Theta \) is compatible with the connected sum \( \# \) and, more generally, let \( V'_3 = o V'_3 + V_3 \), with \( o V'_3 \cap V_3 = M_2 \subset V_3 - \sigma(V_3) \) (a compact 2-manifold). If \( \eta = (\Pi : V'_3 \to V_3) \) is a resolution for \( V_3 \), \( M_2 \) can be lifted to a unique \( M'_2 \subset V'_3 \), and \( \eta \) can be restricted to resolutions \( \eta' \) and \( \eta \). By (a) and the functoriality of \( \Theta \) there exist well-defined embeddings \( M'_2 \times I \subset \partial \Theta_4(\eta) \) and \( M'_2 \times I \subset \partial \Theta_4(\eta) \) (coming from the corresponding \( j \)'s). \( \Theta_4(\eta) \) is obtained by pasting \( \Theta_4(\eta') \) and \( \Theta_4(\eta) \) together along \( M_2 \times I \), and \( j(\eta) \) in a similar way from \( j(\eta'), \theta(\eta) \).

(c) \( \Theta(W \# W) = (D_4, j : W' \to S_3 = \partial D_4, W') \) where \( W', W \) are as above, and \( j \) is the canonical embedding. (It is understood that \( \Theta(\eta) \) is determined only up to "isomorphism.")

This lemma is implicit in [1].
3. We are interested in 3-manifolds with singularities $V_3$, which admit the following description:

We consider $T_{2p}$ and $2p$ differentiable embeddings $\phi^i: S_i \to T_{2p}$ ($i = 1, \ldots, 2p$) such that $\phi^i(S_i) \cap \phi^j(S_i) = \emptyset$ except for $\phi^{2k-1}(S_i) \cap \phi^{2k}(S_i)$ which consists of exactly 2 points, with transversal intersection ($k = 1, \ldots, p$).

We remark that $\phi^{2k-1}(S_i) \cup \phi^{2k}(S_i)$ contains exactly 4 simple circuits of $\partial T_{2p}$ and, for each $k = 1, \ldots, p$, we consider a differentiable embedding $\psi^k: S \to \partial T_{2p} - \bigcup_{i} \phi^i(S_i)$, "parallel" to one of these 4 circuits. We assume that $\psi^i(S_i) \cap \psi^j(S_i) = \emptyset$. We consider some very thin tubular neighborhoods: $\Phi^i$, $\Psi^j: S \times I \to \partial T_{2p} (i = 1, \ldots, 2p; j = 1, \ldots, p)$ of $\phi^i$, $\psi^j$. $S \times I$ is assimilated to $(\partial D^2) \times I \subset \partial (D^2 \times I)$, and hence we can add $3p$ times $D^2 \times I$ along the $\Phi$ and $\Psi$'s, to $T_{2p}$. We get in this way a 3-manifold with singularities

$$V_3 = T_{2p} + (\Phi^1) + \cdots + (\Phi^{2p}) + (\Psi^1) + \cdots + (\Psi^p).$$

We shall consider a resolution $\Pi: V'_3 \to V_3$ which specifies $\Phi^2$, $\Phi^4, \ldots, \Phi^{2p}$.

We shall also consider, for each $k = 1, \ldots, p$ an embedding $\phi^{2k-1}: S_i \to \text{int} T_{2p}$, very close to $\phi^{2k-1}$, and "parallel" to it.

Finally we denote by $\overline{T}_{2p}$ the 3-manifold:

$$\overline{T}_{2p} = T_{2p} + (\partial T_{2p}) \times I$$

where $\partial T_{2p} = \partial T_{2p} \times \emptyset$.

With this we can state

**Theorem 2.** Let $M_3$ be a compact homotopy 3-disk. Then, for some $p = p(M_3)$, the differentiable manifold:

$$M_4^p = (M_3 \# (S_2 \times I) \# \cdots \# (S_2 \times I)(p \text{ times})) \times I$$

$$= (M_3 \times I) \# (S_2 \times D^2) \# \cdots \# (S_2 \times D^2)(p \text{ times})$$

can be described as follows:

There exists a $V_3$ as above, for which the following requirement is fulfilled:

$(\gamma)$ The $2p$ embeddings $S_i \to 2\overline{T}_{2p}$:

$$S_1 \phi^{2k}, \phi^{2k-1} \to T_p \subset \overline{T}_{2p} \subset 2T_{2p} (k = 1, \ldots, p)$$

can be described as follows:

Moreover,

$$\Theta_4(\Pi: V'_3 \to V_3) = M_4^p \text{ (diffeomorphism).}$$
One remarks that the statement $M_3 = D_3$ is equivalent to $M_4^2 = D_4 \# (S_2 \times D_2) \# \cdots \# (S_2 \times D_2)$ ($p$ times) (see [1]). This motivates

**Theorem 3.** Let $V_3$ be the singular 3-manifold described above, but such that the requirement (γ) is replaced by the stronger requirement (Γ):

(Γ) The $2p$ embeddings $S_1 \to \overline{T}_{2p}$:

$$S_1 \xrightarrow{\phi^{2k}, \phi^{2k-1}} T_{2p} \subset \overline{T}_{2p} \quad (k = 1, \ldots, p)$$

are unknotted.

Then:

$$\Theta_4(\Pi: V_4 \to V_3) = D_4 \# (S_2 \times D_2) \# \cdots \# (S_2 \times D_2)(p \text{ times})$$

(diffeomorphism).

**Bibliography**


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