Let $M^n$ be an $n$-dimensional $C^\infty$ manifold and $W^p$ be a $p$-dimensional $C^\infty$ manifold. A $C^\infty$ mapping $f: M^n \to W^p$ is called a $k$-mersion if its rank is greater than or equal to $k$ everywhere. The set of $k$-mersions, endowed with the $C^1$ topology, is denoted $R(M^n, W^p; k)$. A $k$-regular homotopy between $k$-mersions $f$ and $g$ is a continuous mapping $F: I \to R(M^n, W^p; k)$ such that $F(0) = f$ and $F(1) = g$.

A $k$-bundle map, $\psi: TM^n \to TW^p$ between the tangent spaces of $M^n$ and $W^p$ is a continuous fibre preserving mapping such that the restriction of $\psi$ to any fibre is a linear map of rank at least $k$. The space of $k$-bundle maps with the compact open topology is denoted $T(M^n, W^p; k)$.

An $n$-mersion is an immersion, and an $n$-regular homotopy is usually called a regular homotopy. In 1958 and 1959, Smale [4], [5] published papers classifying immersions of spheres in Euclidean spaces. Smale proved that if $n < p$, the regular homotopy classes of immersions of $S^n$ in $E^p$ are in one to one correspondence with the homotopy classes of sections of $S^n$ into the bundle associated with $TS^n$ whose fibre is the Stiefel manifold $V_{p,n}$ of $n$ frames in $p$-dimensional Euclidean space. Smale obtained this classification by proving a stronger result, namely, that the map $d: R(S^n, E^p; n) \to T(S^n, E^p; n)$ defined by $d(f) = df$ is a weak homotopy equivalence if $n < p$. His proof was based on the diagram

$$
\begin{align*}
R(S^n, E^p; n) \xrightarrow{d} & \to T(S^n, E^p; n) \\
\downarrow i^* & \downarrow j^* \\
R(D^n, E^p; n) \xrightarrow{d} & \to T(D^n, E^p; n)
\end{align*}
$$

where $D^n$ is identified with a hemisphere of $S^n$, and $i^*$ and $j^*$ are restriction maps. The main step in the proof consists of showing that $i^*$

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1 This work was performed in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Cornell University, 1967. I wish to thank Professor R. Szczarba of Yale University, under whose direction this work was done.
and \( j^* \) are fiberings (i.e., have covering homotopy property). Then it is easily shown that the \( d \) of the bottom row is a weak homotopy equivalence and that \( d \) restricted to a fibre of \( i^* \) is a weak homotopy equivalence. It follows immediately from the homotopy sequence of a bundle and the five lemma that the \( d \) of the top row is a weak homotopy equivalence.

In 1959, Hirsch [1] extended this result to the case of immersions \( R(M^n, W^p; n) \) of a \( C^\infty \) manifold in another, with \( \dim M^n < \dim W^p \), (i.e. \( n < p \)) and \( \partial W^p = \emptyset \). Poenaru's exposition of this result [3] was the basis of Phillips' thesis in 1965 (published as [2]) which stated that if \( M^n \) has no compact components with empty boundary and \( \partial W^p = \emptyset \), then \( d: R(M^n, W^p; p) \to T(M^n, W^p; p) \) is a weak homotopy equivalence. Phillips called the maps whose rank equalled the dimension of the image space "submersions."

Poenaru's exposition also is the basis of the generalization given here.

**Theorem 1.** Let \( M^n \) and \( W^p \) be \( C^\infty \) manifolds with \( \partial W^p = \emptyset \). The mapping \( d: R(M^n, W^p; k) \to T(M^n, W^p; k) \) defined by \( d(f) = df \) is a weak homotopy equivalence if either

(a) \( M^n \) has no compact components with empty boundary, or

(b) \( k < p \).

**Corollary 1.** If condition (a) or condition (b) of Theorem 1 is satisfied, the \( k \)-regular homotopy classes of \( k \)-mersions of \( M^n \) in \( W^p \) are in one to one correspondence with the homotopy classes of \( k \)-bundle maps of \( TM^n \) in \( TW^p \).

Denote by \( M^\ast(p, n; k) \) the set of \( p \times n \) matrices of rank at least \( k \).

**Corollary 2.** Suppose condition (a) or condition (b) is satisfied. The \( k \)-regular homotopy classes of \( k \)-mersions of \( M^n \) in \( W^p \) are in one to one correspondence with the homotopy classes of sections of \( M^n \) into the bundle associated with \( TM^n \) whose fibre is \( M^\ast(p, n; k) \).

As an application of the above, it can be shown that if \( p \geq (3/2)(n - 1) \), there exists an \( n - 1 \) mersion of \( M^n \) in \( R^p \).

The proof of Theorem 1 uses a filtration of \( M^n \)

\[
D^n = U_0^n \subset U_1^n \subset U_2^n \cdots \subset M^n
\]

where \( U_i \) is obtained from \( U_{i-1} \) by adding a handle, essentially. The simple scheme of (1) is replaced by
$R(M^n, W^p; k) \xrightarrow{d} T(M^n, W^p; k)$
\[
\begin{array}{c}
n \downarrow \\
\vdots \\
n \downarrow \\
R(U^n_2, W^p; k) \xrightarrow{d} T(U^n_2, W^p; k) \downarrow i_*^* \\
\downarrow j_*^*
\end{array}
\]

$R(U^n_1, W^p; k) \xrightarrow{d} T(U^n_1, W^p; k) \downarrow i_*^* \downarrow j_*^*$

$R(D^n, W^p; k) \xrightarrow{d} T(D^n, W^p; k)$

It is easy to show that the $d$ of the bottom row is a weak homotopy equivalence, and that all of the $j^*$ maps are fiberings. The main step in the proof is in showing that the $i^*$ maps are fiberings from which it also easily follows that $d$ restricted to a fibre is a weak homotopy equivalence. The rough outlines of the proof that the $i^*$ maps are fiberings are as follows. Let $V^n$ be obtained from $U^n$ by adding a handle of index $\lambda$. Denote $I^m = ([0, 1])^m$, and $I^{m-1}([0, 1])^{m-1} \times \{0\}$. The map $i^*$ has the covering homotopy property if, given continuous maps $g: I^m \to R(U^n, W^p; k)$ and $G: I^{m-1} \to R(V^n, W^p; k)$ such that $G(q) \mid U^n = g(q)$ when $q \in I^{m-1}$, there is a mapping $\overline{G}: I^m \to R(V^n, W^p; k)$ with

$$
\overline{G}(q) = G(q), \quad q \in I^{m-1}, \quad \overline{G}(q) \mid U^n = g(q), \quad q \in I^m.
$$

(3) $\overline{G}(q)$ is an extension of $g(q)$ over $V$, so $\overline{G}$ can be viewed as a continuously varying set of extensions of $g$ over $V$ which agrees with $G$ on $I^{m-1}$.

If, for each $f \in R(U^n, W^p; k)$ there is a neighborhood $\eta(f)$ for which $i^* \mid \eta(f) = (i^* \mid i^{m-1}(\eta(f)))$ has the covering homotopy property, it is very easy to show that $i^*$ has the covering homotopy property. Thus, given $f \in R(U^n, W^p; k)$ it suffices to find a neighborhood $\eta$ of $f$ such that if $g$ and $G$ map into $\eta$ and $i^{m-1}(\eta)$ respectively, an extension $\overline{G}$ satisfying (3) can always be constructed.

As a first step, it is not difficult to show that there is a neighborhood $\eta$ of $f$ whose elements can be factored through automorphisms of some compact $(n+\rho)$-dimensional manifold $C$. In fact, there is a compact neighborhood $N^n$ of $U^n$ in $V^n$, an embedding $s: N^n \to C$, a differentiable map $P: C \to W^p$, and a continuous mapping $\nu: \eta \to \text{Aut } C$ (the automorphisms of $C$ which equal the identity near $\partial C$) so that $P \circ \nu(h) \circ s \mid U = h$, $h \in \eta$. 

$P \circ \nu(h) \circ s \mid U = h, \ h \in \eta.$
Note that $P \circ v(h) \circ s$ is an extension of $h$ to $N^n$, and if $g: I^m \to \eta$, then $P \circ v(g(q)) \circ s$ is a continuously varying set of extensions of $g$ over $N$. It turns out that $v$ can be modified so that $P \circ v(g(q)) \circ s = G(q)|_N$ if $q \in I^{m-1}$. Thus the factorization (4) enables a lift to be constructed over a part of the handle, at least.

Near $\partial C$, $v(h)$ is always the identity. The second part of the construction uses this fact. Let $\tilde{N}$ be the boundary of $N^n$ in $V^n$, i.e., $\tilde{N} = (\partial N^n - \partial V^n)^-$. Since $N^n$ is embedded in $C$, we can assume $N^n \subset C$. If $N^n$ can be deformed in $C$ through embeddings $\xi_t, t \in [0, 1]$, so that $U^n$ is always left fixed, but $\tilde{N}$ is carried out to $\partial C$ (i.e., $\xi_t(\tilde{N}) \subset \partial C$), then for $x$ near $\tilde{N}$, $P \circ v(g(q)) \circ \xi_t(x) = P \circ \xi_t(x)$ for all $q \in I^m$. Using this fact, it is not too difficult to piece together an explicit formula defining $G$. The only use of the hypotheses, condition (a) or condition (b), is in proving the existence of such a deformation $\xi_t$. This turns out to be simple in case (a) but case (b) is quite complicated when the index of the handle is $n$.

The details of the proof and a number of applications will be presented in a forthcoming paper.

BIBLIOGRAPHY

5. ———, Classification of immersions of spheres in Euclidean space, Ann. of Math. (2) 69 (1959), 327–344.

CORNELL UNIVERSITY