FLATTENING A SUBMANIFOLD IN CODIMENSIONS ONE AND TWO

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Communicated by O. G. Harrold, November 3, 1967

Let $M$ and $N$ be manifolds with $M \subset \text{Int } N$, and assume that $M - X$ is locally flat in $N$, where $X$ is some subset of $M$. We are interested in finding conditions (intrinsic, placement, dimensional, etc.) which, when placed on $X$, imply that $M$ is locally flat in $N$. Extremely useful and satisfying answers are provided by Bryant and Seebeck in [2], assuming that $\dim N - \dim M \geq 3$. We announce here a method for deducing local versions of Corollary 1.1 of [2] in codimensions one and two.

DEFINITIONS. If $M$ is a manifold, a collaring of $\text{Bd } M$ in $M$ is an embedding $\lambda$ of $\text{Bd } M \times [0, \infty)$ into $M$ such that $\lambda(x, 0) = x$ for each $x$ in $\text{Bd } M$. We use $\mathbb{R}^n$ to denote euclidean $n$-space, $B^n$ the closed unit ball in $\mathbb{R}^n$.

THEOREM. For integers $0 \leq k < m \leq n$, let $D$ be an $m$-cell in $\mathbb{R}^n$ and let $E$ be a $k$-cell in $\text{Bd } D$. Assume that the following condition is satisfied:

$D - E$ is locally flat in $\mathbb{R}^n$ and $E$ is locally flat in $\text{Bd } D$.

Then $(\mathbb{R}^n, D) \approx (\mathbb{R}^n, B^m)$ if and only if $E(\lambda|I)$ is locally flat in $\mathbb{R}^n$ for some collaring $\lambda$ of $\text{Bd } D$ in $D$.

The proof of this theorem is similar to the proof of Theorem 4.2 of [7]. Theorem 4.1 of [7] must be used more carefully to replace Corollary 3.2 of [7].

A detailed proof of the above theorem, together with applications and generalizations, will appear elsewhere. We present below the immediate implications of [2]. (Actually, in an earlier paper which is in press, Bryant and Seebeck prove a local form of Corollary 1.1 of [2] which is enough to yield the following applications.)

REMARK. There are no dimensional restrictions (other than $0 \leq k < m \leq n$) in the above Theorem.

APPLICATION 1. Let $D$ be an $m$-cell in $\mathbb{R}^n$, and let $E$ be a $k$-cell in $\text{Bd } D$. Assume that

$D - E$ and $E$ are locally flat in $\mathbb{R}^n$, and $E$ is locally flat in $\text{Bd } D$.

If $k \leq n - 4$ then $(\mathbb{R}^n, D) \approx (\mathbb{R}^n, B^m)$.

1 Supported by the National Science Foundation and a Alfred P. Sloan fellowship.

2 Supported by the National Science Foundation.
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PROOF. Let \( \lambda \) be a collaring of \( \text{Bd } D \) in \( D \). If \( n \geq 4 \) and \( k = 0 \), then \( \lambda(E \times I) \) is locally flat in \( R^n \) by [3]. If \( n \geq 5 \), \( \lambda(E \times I) \) is locally flat in \( R^n \) by Corollary 1.1 of [2]. In either case the result follows from our Theorem.

REMARKS. 1. The analogue of Application 1 for \( k = n - 3 \geq 0 \) is false; the Theorem may still be applied to specific cases, however.

2. There are no restrictions on \( m \) and \( n \) in Application 1.

DEFINITION. Let \( \beta(n) \) denote the following conjecture: If \( D_1 \) and \( D_2 \) are locally flat \((n-1)\)-cells in \( R^n \) such that \( D_1 \cap D_2 = \text{Bd } D_1 \cap \text{Bd } D_2 \) is an \((n-2)\)-cell whose boundary is locally flat in both \( \text{Bd } D_1 \) and \( \text{Bd } D_2 \), then \( D_1 \cup D_2 \) is locally flat in \( R^n \).

Conjecture \( \beta(3) \) is proved in [5]. A proof of \( \beta(n) \), \( n \geq 5 \), is announced and outlined by Černavskii in [4]. \( \beta(4) \) has recently been proved by Černavskii and by R. C. Kirby.

APPLICATION 2. Let \( D \) be an \((n-1)\)-cell in \( R^n \), and let \( E \) be a \( k \)-cell in \( D \). Assume that

\[
(D, E) \text{ is a proper locally flat cell pair, and }
\]

\[
D - E \text{ and } E \text{ are locally flat in } R^n.
\]

If \( k \leq n - 4 \) then \((R^n, D) \approx (R^n, B^{n-1})\).

PROOF. Let \( f: (B^{n-1}, B^k) \approx (D, E) \) be a homeomorphism. (See [6].) Let \( D_1 = f(B^{n-1}) \) and \( D_2 = f(B^{n-1}) \). By Application 1, \( D_1 \) and \( D_2 \) are locally flat. By \( \beta(n) \), \( D \) is locally flat.

REMARKS. 1. The analogue of Application 2, with \( D \) an \((n-2)\)-cell, is false for \( n \geq 3 \).

2. The Theorem and Applications can be applied locally to embeddings of manifolds.

REFERENCES


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