INTERIORITY OF A HOLOMORPHIC MAPPING ON THE SET OF ITS EXCEPTIONAL POINTS

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I. Introduction. A mapping $f: A \rightarrow B$ is said to be interior (or open) if for every open subset $U \subseteq A$, $f(U)$ is an open subset of $B$; it is said to be interior at a point $a \in A$ (or locally interior at $a \in A$) if for every open subset $U \subseteq A$ containing $a$, $f(a)$ is an interior point of $f(U)$. Clearly a mapping is interior if and only if it is locally interior everywhere on its domain of definition.

The result contained in this note is about the local interiority property of a holomorphic mapping on the set of its exceptional points. We shall restrict our attention to holomorphic mappings $f = (f_1(x), \ldots, f_n(x)): D \rightarrow \mathbb{C}^n$ where $D$ is a domain (open connected set) in $\mathbb{C}^n$. $\mathbb{C}^n = \mathbb{C}^1 \times \cdots \times \mathbb{C}^1$ where $\mathbb{C}^1$ is the extended plane of each one of the complex variables $x_i$. $f$ is said to be holomorphic when each one of the functions $f_i$ is holomorphic on $D$. Let $J(x)$ be the value of the Jacobian of $f$ at $x \in D$.

The set $E$ of exceptional points of $f$ is by definition $E = \{ x \in \Omega | f \text{ is interior at } x \}$.

II. Result. We recall that if $a \in E$, $f$ is interior at $a$. In fact, if $a \in E$ and $J(a) \neq 0$, the property follows immediately from the inverse function theorem ($f$ is a local homeomorphism); if $a \in E$ and $J(a) = 0$, it follows from a theorem of Osgood [1] ($f$ maps finitely-to-one sufficiently small neighborhoods of $a$ onto neighborhoods of $b = f(a)$).

Our result pertains to the case $a \in E$:

**THEOREM.** Let $f: D \rightarrow \mathbb{C}^n$, $D \subseteq \mathbb{C}^n$, be a holomorphic mapping and let $E$ be the set of exceptional points of $f$, then the subset $E_0$ in $E$ such that $E_0 = \{ x \in E | f \text{ is interior at } x \}$ is either the empty set or a set of isolated points.

**PROOF.** If $E$ is empty, $f$ is everywhere interior in $D$ as shown above. If $f$ is degenerate, i.e., $J(x) \equiv 0$, it is not difficult to show that $E = D$ and $E_0 = \{ \emptyset \}$.

Let then $f$ be not degenerate and $E$ not empty. H. Cartan [2] proved that $E$ is an analytic set and $E \subseteq W = \{ x \in D | J(x) = 0 \}$. Com-

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plex-dimension \((W) = n - 1\) and complex-dimension \((W' = f(W)) \leq n - 1\). Let 
\[ S = \{ S_1, \ldots, S_r \} \subset E \]
be the set (finite) of irreducible local analytic varieties passing through a given arbitrary point \(a \in E\) and let 
\[ V = \{ V_1, \ldots, V_s \} \]
be the set (finite) of irreducible subvarieties in \(S\) which are associated with \(a\), meaning that 
\[ f(V) = f(a) = b = (b_1, \ldots, b_n). \]
Now we consider any one-complex-dimensional analytic plane \(\Pi\) passing through \(b\) and not contained in \(W'\). Let 
\[ \Pi = \{ y \in \mathbb{C}^n \mid (y_1 - b_1)/\alpha_1 = \cdots = (y_n - b_n)/\alpha_n \} \]
where \(\alpha_1, \ldots, \alpha_n\) are complex constants, be that plane. Obviously 
\[ f^{-1}(\Pi) = \{ x \in D \mid (f_1(x) - b_1)/\alpha_1 = \cdots = (f_n(x) - b_n)/\alpha_n \}. \]
This is an analytic set, consequently, \([3]\), locally at the given point \(a \in E\) it consists of a finite set of irreducible analytic varieties which will be called \(\theta\). Clearly, \(\theta \supseteq V\) since \(f(V) = b\) and \(b \in \Pi\).

\textbf{Case 1.} \(\theta = V\), then \(f\) is not locally interior at \(a\). Indeed, if \(N_a \subset D\) is a sufficiently small neighborhood of \(a\), \(b\) will be the only point in \(\Pi\) contained in \(f(N_a)\); this proves that \(b = f(a)\) is on the boundary of \((N_a)\).

\textbf{Case 2.} \(\theta \supset V\). This means that \(\theta = \{ V, \theta_1, \ldots, \theta_p \} \) where \(\theta_1, \ldots, \theta_p\) are the irreducible analytic varieties in the local decomposition of \(f^{-1}(\Pi)\) which are not contained in \(V\). Since \(\Pi\) is not contained in \(W'\), none of the \(\theta_i\) is contained in \(W\). Hence, each one of the \(\theta_i\) being mapped under \(f\) into \(\Pi\) is itself of complex-dimension 1. This proves that the intersection of the \(\theta_i\) with \(E\) is a set \(E^* \subset E\) which consists of isolated points. In order for \(f\) to be locally interior at \(a\) it is necessary that for every \(\Pi\) defined as above there exist varieties \(\theta_i\). Hence, the set \(E_0 \subset E\) of points where \(f\) is locally interior certainly satisfies \(E_0 \subset E^*\) (\(E^*\) was defined for a single \(\Pi\)) and therefore \(E_0\) contains at most isolated points. Q.E.D.

As an immediate corollary we obtain a result proved by R. Remmert \([4]\).

\textbf{Corollary.} A holomorphic mapping \(f: D \to \mathbb{C}^n, D \subset \mathbb{C}^n\), is interior if and only if \(E\) is the empty set.

\textbf{III. Examples.} In order to show that the two possibilities for \(E_0\) which were mentioned in the Theorem can actually occur, we give the two following examples.

\textbf{Example 1.} \(f = (y_1 = x_1x_2, y_2 = x_2): \mathbb{C}^2 \to \mathbb{C}^2\). Here \(J(x) = x_2, \quad E = W = \{ x \in \mathbb{C}^2 \mid x_2 = 0, \quad x_1\ \text{arbitrary} \}, \quad W' = f(E) = \{ 0' = (y_1 = y_2 = 0) \}. \) It is clear that the set \(\Pi = \{ 0' \}, \) where \(\Pi = \{ y \in \mathbb{C}^2 \mid y_2 = 0, \quad y_1\ \text{arbitrary} \},\) is not in the range of \(f\). Thus, \(\forall a \in E\) and any open set \(N_a \subset \mathbb{C}^2 \exists a \in N_a,\)
$0' = f(a)$ is on the boundary of $f(N_a)$. This proves that $E_0 = \{ \emptyset \}$.

**EXAMPLE 2.** $f = (y_1 = x_1(x_3-x_1), y_2 = x_3(x_2 + x_3), y_3 = x_1x_2x_3) : C^3 \to C^3$. Here $E = \{ x \in C^3 | x_1 = 0, x_2 \text{ and } x_3 \text{ arbitrary} \}$. We shall show that $f$ is locally interior at $0 \in E$, $0 = (x_1 = x_2 = x_3 = 0)$, by proving that for arbitrarily small $\delta_1 > 0$, $\exists \delta_1(\delta_1) > 0 \exists y \in \text{boundary}(S')$ and $0 < \delta < \delta_1$, $\exists x \in S \exists f(x) = y$, where $S$ and $S'$ are open hyperspheres, respectively, centered at 0 and $0'$ with radius $\delta_1$ and $\delta$.

From the equations defining $f$ we can derive:

1. $x_1^4 - x_1^2(y_2 - 2y_1) + x_1y_3 + y_1(y_1 - y_2) = 0$,
2. $x_2 = (y_2 - y_1)/x_1 - x_1$,
3. $x_3 = y_1/x_1 + x_1$.

Let us consider a surface $S = \{ y | |y_1|^2 + |y_2|^2 + |y_3|^2 = \epsilon^2 \}$ where $0 < \epsilon \ll 1$. Our first step is to find a common upper bound for the roots $x_i$, $i = 1, \cdots, 4$, of equation (1) when $y \in S$. We can write (1) as

$$x_1 \leq (y_3 - 2y_1)/2 \pm (y_2^2/4 - x_2y_3)^{1/2}.$$ 

For $y \in S$, we obtain from (1')

$$|x_1|^2 < \frac{3\epsilon^2}{2} + \left(\frac{\epsilon^2}{4} + |x_1| \epsilon^2\right)^{1/2} < \frac{3\epsilon^2}{2} + \left(\frac{\epsilon^2}{4}\right)^{1/2} + (|x_1| \epsilon^2)^{1/2} = 2\epsilon^2 + (|x_1| \epsilon^2)^{1/2}.$$ 

Since $\epsilon \ll 1$, it is not difficult to see that this inequality holds for

1. $|x_1| < \epsilon + o(\epsilon^2)$ where $o(\epsilon^2)$ is of the order of $\epsilon^2$ when $\epsilon \to 0$.

Now let $x_i^m$ be one of the four roots $x_i$ whose absolute value is larger or equal to the absolute value of all the others. We want to find a lower bound for $x_i^m$. To that purpose we introduced the following symmetric functions of the $x_i$, obtained from (1):

$$s_4 = x_1x_2x_3x_4 = y_1(y_1 - y_2),$$
$$s_3 = x_1x_2x_3 + \cdots + x_4x_3x_1 = (\text{total of 4 terms}) = - y_3,$$
$$s_2 = x_1^2 + \cdots + x_4^2 = (\text{total of 6 terms}) = y_3 - 2y_1.$$

Clearly:

$$|x_i^m| \geq |s_i|^{1/4} \geq |y_1|^{1/4} |y_1| - |y_2| |y_1|^{1/4},$$
$$|x_i^m| \geq |s_2/4|^{1/3} = |y_3/4|^{1/3},$$
$$|x_i^m| \geq |s_2/6|^{1/2} \geq \left|\left( y_2 - 2y_1 \right) / 6 \right|^{1/2}.$$
Therefore
\[
\left| x_1^m \right| \geq \frac{\left| y_1 \right|^{1/4} \left| y_1 \right| - \left| y_2 \right|^{1/4} + \left| y_2/4 \right|^{1/4} + \left( \left| y_2 \right| - 2 \left| y_1 \right| \right)/6}{3} \cdot
\]
\( \forall y \in \sigma \), it follows from this last inequality that \( \left| x_1^m \right| > \varepsilon^{3/2}/9 \). Hence, recalling (4), we have
\begin{equation}
(5) \quad \varepsilon^{3/2}/9 < \left| x_1^m \right| < \varepsilon + o(\varepsilon^2).
\end{equation}
Finally from (2), (3) and using (5) we obtain:
\[
\left| x_2^m \right| \leq \frac{\left| y_2 \right| + \left| y_1 \right| + \left| x_1^m \right|^2}{\left| x_1^m \right|} < \frac{2\varepsilon \varepsilon + \varepsilon^2 + o(\varepsilon^3)}{\varepsilon^{3/2}/9} = 9\varepsilon^{1/2} + o(\varepsilon^{3/2}),
\]
\[
\left| x_3^m \right| \leq \frac{\left| y_1 \right| + \left| x_1^m \right|^2}{\left| x_1^m \right|} < \frac{\varepsilon \varepsilon + \varepsilon^2 + o(\varepsilon^3)}{\varepsilon^{3/2}/9} = 9\varepsilon^{1/2} + o(\varepsilon^{3/2}).
\]
If \( \varepsilon \) is taken to be sufficiently small, then certainly
\[
\left| x_1^m \right| < \varepsilon + o(\varepsilon^2) < 10\varepsilon^{1/2}, \quad \left| x_2^m \right| < 9\varepsilon^{1/2} + o(\varepsilon^{3/2}) < 10\varepsilon^{1/2},
\]
\[
\left| x_3^m \right| < 9\varepsilon^{1/2} + o(\varepsilon^{3/2}) < 10\varepsilon^{1/2}.
\]
In order to complete the required proof it is enough to put
\[
\varepsilon_1 = 10\varepsilon^{1/2} \quad \text{and} \quad \delta_1 = \frac{3}{2} = 10^{-4} \times \varepsilon_1.
\]
By using arguments similar to those given in the proof of the Theorem, it is possible to show that \( \forall a \in E \) and \( a \neq 0, f \) is not interior at \( a \). Thus \( E_0 = \{ 0 \} \neq \{ \emptyset \} \).

**REFERENCES**


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