Artin's celebrated conjecture on primitive roots (Artin [1, p. viii], Hasse [2], Hooley [3]) suggests the following

**Conjecture.** Let $S'$ be a set of rational primes. For each $q \in S'$, let $L_q$ be an algebraic number field of degree $n(q)$. For every square-free integer $k$, divisible only by primes of $S$, define $L_k$ to be the composite of all $L_q$, $q \mid k$, and denote $n(k) = \deg(L_k/Q)$. Assume that $\sum_k 1/n(k)$ converges, where the sum is over those $k$ for which $L_k$ is defined. Then the natural density of the set $P$ of all primes $p$ which do not split completely in each $L_q$ exists and has the value $\sum_k \mu(k)/n(k)$, where $\mu$ is the M"obius function and the term $k = 1$ has been included with $\mu(1) = 1$.

If $S = \{\text{all rational primes}\}$, $L_q = \mathcal{O}(\xi_q, aq^\ell)$, $a \in \mathbb{Z}$, $\xi_q$ a primitive $q$th root of 1, then the conjecture is equivalent to Artin's conjecture. If $S$ is a finite set, then the conjecture is easily verifiable using the prime ideal theorem. For $S = \{\text{all rational primes}\}$, $L_q = \mathcal{O}(\xi_q)$, the conjecture has been proved by Knobloch [4] (for $r = 2$ and only for Dirichlet densities) and by Mirsky [5].

We have proved the following theorems, whose proofs will appear elsewhere.

**Theorem 1.** Let there exist a finite set $S_0 \subset S$ such that $L_q \supset \mathcal{O}(\xi_q)$ for $q \in S - S_0$, and $L_q/Q$ is normal for all $q \in S$. Then the conjecture is true.

**Theorem 2.** Suppose that for each finite subset $S_0 \subset S$ there exists a family of algebraic number fields $\{L_q\}_{q \in S}$ such that

1. $L_q = L'_q$ for $q \in S_0$,
2. $L_q' \subset L_q$ for all $q \in S$,
3. $L_q' \neq Q$ for all $q \in S$,
4. the conjecture is true for $\{L'_q\}_{q \in S}$.

Then the conjecture is true for $\{L_q\}_{q \in S}$.

**Theorem 3.** If the density $d(P)$ of $P$ exists, then

$$d(P) \leq \sum_k \mu(k)/n(k).$$

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1 Research partially supported by Air Force Office of Scientific Research Grant No. SAR/F-44620-67.
Theorem 1 is the main result. Theorems 2 and 3 are elementary in character. The proof of Theorem 1 is divided into two parts: First, it is shown that one may compute the number of primes \( p \leq x \) in \( P \) by computing the number of primes \( p \leq x \) which do not split completely in \( L_q \) for all "sufficiently small \( q \)" where the upper bound for \( q \) is a function of \( x \). Computing this latter quantity is reduced to computing the number of prime ideals of \( L_k \) which have norm \( \leq x \), for all "sufficiently small \( k \)". The prime ideal theorem asserts that this latter quantity is asymptotically equal to \( x / \log x \). But the error term will, in general, depend on \( L_k \). The second part of the proof consists in showing that by restricting \( k \) to be "sufficiently small" one can choose the error term to be independent of \( k \). This result constitutes a generalization of the uniform prime number theorem of Siegel and Walfisz (Prachar [6, p. 144]) for primes in arithmetic progressions. In fact, we can prove our theorem in a very general setting, which, although not required for the proofs of Theorems 1–3, seems interesting for its own sake.

Let \( K \) be a normal algebraic number field of degree \( n \) and discriminant \( d \). Let \( \alpha \rightarrow \alpha^{(j)} (1 \leq j \leq n) \) be the embeddings of \( K \) in the complex numbers \( C \), ordered so that the first \( r_1 \) are real and the \( j \)th and \((j+r)\)th \((r_1+1 \leq j \leq r_1+r_2)\) constitute a pair of complex-conjugate embeddings. Let

\[
n_j = 1, \quad 1 \leq j \leq r_1
\]
\[
= 2, \quad r_1 + 1 \leq r_1 + r_2.
\]

For \( \alpha \in K^* = K - \{0\} \), let \( \alpha \equiv 1 \pmod* \alpha \) mean that \( \alpha \) is multiplicatively congruent to 1 modulo the \( K \)-ideal \( \alpha \). For \( \alpha \in K^* \), denote by \((\alpha)\) the \( K \)-ideal generated by \( \alpha \). Let \( \chi \) be a grossencharacter of \( K \) having conductor \( f \). For \( \alpha \equiv 1 \pmod* f \), let

\[
\chi((\alpha)) = \prod_{j=1}^{r_1+r_2} \left( \frac{\alpha^{(j)}}{|\alpha^{(j)}|} \right)^{n_j} |\alpha^{(j)}|^{\nu_{n_j, \phi_j}}
\]

where \( m_j = 0, 1 \) and \( \phi_j \in \mathbb{R} \) are normalized so that \( \sum_{j=1}^{r_1+r_2} n_j \phi_j = 0 \). Let

\[
\pi(x, K, \chi) = \sum_{N(p) = A \chi(p) n_j \phi_j = 0} \chi(p)
\]

where the sum is over primes \( p \) of \( K \). For \( A > 0 \), define \( B(A) = \{ \chi \) a grossencharacter of \( K \) | \( |\phi_j| \leq A, 1 \leq j \leq r_1 + r_2 \} \). Then we have the following generalization of the Siegel-Walfisz theorem:

**Theorem 4.** Let \( A > 0, \varepsilon > 0 \) be given. Then there exists a positive constant \( c = c(A, \varepsilon) \), not depending on \( K, n, d \), or \( \chi \) such that for \( \chi \in B(A) \),
\[ \pi(x, K, \chi) = E(x) \ln x + O(Dx \log^2 x \exp\{ -c_n(\log x)^{1/2}/D \}), \quad x \to \infty \]

where the 0-term constant does not depend on \( K, \chi, n \) or \( d \) and

\[ E(x) = 0, \quad \chi \neq \text{the trivial grossencharacter} \]
\[ = 1, \quad \chi = \text{the trivial grossencharacter}, \]

\[ \text{li } x = \int_2^x \frac{dy}{\log y}, \]

\[ D = n^4[|d| N(f)]^s e^{-n}. \]

**BIBLIOGRAPHY**


**Yale University**