ON THE EXISTENCE OF EXCEPTIONAL FIELD EXTENSIONS

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Let $F$ be a field of characteristic $p \neq 0$ and let $K$ be an algebraic field extension of $F$. Let $K_i$ denote the subfield of $K$ of elements purely inseparable over $F$, $K_s$ the subfield of separable elements, and $K^n$ the normal closure of $K/F$. We say that $K/F$ splits if $K = K_iK_s$ and following Reid’s terminology in [2], $K$ is called an exceptional extension of $F$ provided $K_i = F$ and $K_s \neq K$.

**Lemma 1.** $K/F$ splits if and only if $K_i = (K^n)_i$.

**Proof.** If $K/F$ splits it follows easily that $K_i = (K^n)_i$. Conversely assume that $K_i = (K^n)_i$. Then $K^n/K$ is separable normal and hence a Galois extension. Since a normal extension splits we have $K^n = (K^n)_i(K^n)_s$ and if $a \in K$, $a = \sum a_a e_a$ with $a_a \in (K^n)_s$ and $\{e_a\}$ a linearly independent set of elements of $(K^n)_i = K_i$ over $F$. If $\sigma$ is an automorphism of $K^n/K$ then $\sigma(a) = a$ implies that $\sum (\sigma(a_a) - a_a)e_a = 0$. But $K_i$ and $(K^n)_s$ are linearly disjoint over $F$ so that $\{e_a\}$ is linearly independent over $(K^n)_i$. Hence $\sigma(a_a) = a_a$ and we have $a_a \in K \cap (K^n)_s = K_s$. Thus $K = K_sK_i$.

**Theorem 2.** If $K/F$ is a simple extension then $K/F$ splits if and only if $K^n/F$ is simple.

**Proof.** If $K/F$ splits then by Lemma 1, $K_i = (K^n)_i$ and it is clear that $K^n/F$ is also simple.

If $K^n/F$ is simple then $K/F$ and $(K^n)_i/F$ are simple. Let $f(X)$ be the minimum polynomial of $t$ over $F$, where $t$ is chosen such that $K = F(t)$. Then $K^n$ is the splitting field of $f(X)$ and we have

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\begin{align*}
(a) \quad \exp f(X) &= \exp((K^n)_i), \\
(b) \quad \exp f(X) &= [K: K_s].
\end{align*}
\]

Since $(K^n)_i/F$ is simple it follows that $\exp f(X) = [(K^n)_i : F]$ [3, pp. 120–123]. Hence $[K: K_s] = [(K^n)_iK_s : K_s]$ and since $K \subseteq (K^n)_s K_s$ we have $(K^n)_i K_s = K$ and $(K^n)_i = K_i$. By Lemma 1, $K/F$ splits.

Our next lemma gives a method for constructing exceptional field extensions.

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Lemma 3. Let \( a, b, \) and \( s \) be elements of an algebraic extension field of \( F \) with \( a \) and \( b \) purely inseparable over \( F \), \( s \) separable over \( F \) and not in \( F \). Let \( t=a+bs \) and \( K=F(t) \). Then \( F(a, b)=(K^n)_i \) and \( F(a, b)/F \) is generated by the coefficients of the minimum polynomial for \( t \) over \( F(a, b) \).

Proof. Let \( s=s_1, s_2, \ldots, s_n \) be a complete set of conjugates of \( s \) over \( F \) and let \( t_i=a+bs_i \). If \( e \) is a nonnegative integer such that \( a^s, b^s \in F \), then \( F(t_i^e)=F(s_i^e)=F(s_i) \). Hence \( F(s_1, \ldots, s_n) \subseteq F(t_1, \ldots, t_n) \). Also \( b=(t_1-t_2)(s_1-s_2)^{-1} \) so that \( b \), and hence \( a \), are in \( F(t_1, \ldots, t_n) \). It follows that \( F(t_1, \ldots, t_n)=F(a, b) \otimes F(s_1, \ldots, s_n) \). And since the \( t_i \) are conjugates over \( F \), we have \( F(t_1, \ldots, t_n)=K^n \) and \( F(a, b)=(K^n)_i \) [1, p. 50]. The minimum polynomial for \( t \) over \( F(a, b) \) is \( g=\prod_{i=1}^{n}(X-t_i) \). If \( F_0 \) is the subfield of \( F(a, b) \) obtained by adjoining the coefficients of \( g \) to \( F \), then \( F_0/F \) is purely inseparable and \( K^n/F_0 \) is separable. Therefore, \( F_0=(K^n)_i=F(a, b) \).

Remark 4. Reid calls a separable field extension \( E/F \) realizable if there exists an exceptional extension \( K/F \) with \( E=K^n \) [2]. Using Lemma 3 we can show that when \( F/F^p \) is not simple then any proper separable extension of \( F \) is realizable.

Theorem 5. Let \( K/F \) be normal and inseparable, but not purely inseparable. Then \( K/F \) is simple if and only if every subextension of \( K/F \) splits.

Proof. If \( K/F \) is simple and \( E \) is an intermediate field then we can take \( E^n \subseteq K \). Hence \( E^n/F \) is simple and by Theorem 2, \( E/F \) splits. Conversely if \( K/F \) is not simple then \( K_i/F \) is not simple. Hence there exist \( a, b \in K_i \) such that \( F(a, b)/F \) is not simple. We choose \( s \in K_i - F \) and set \( t=a+bs \). If \( E=F(t) \) then by Lemma 3, \( F(a, b) \subseteq E^n \) so that \( E^n/F \) is not simple. Hence by Theorem 2, \( E/F \) does not split.

Our next result gives necessary and sufficient conditions that a given normal inseparable extension \( K/F \) contain intermediate fields which are exceptional over \( F \).

Theorem 6. Let \( K/F \) be normal and inseparable but not purely inseparable. Let \( E \) be the maximal purely inseparable subfield of \( K/F \) of exponent one. Then \( E/F \) is simple if and only if \( K/F \) contains no exceptional subextensions.

Proof. If \( K/F \) contains an exceptional subextension then \( K \) contains an element \( t \) such that \( F(t)/F \) is exceptional of exponent one.

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1 The proof of Lemma 3 indicated here is that of H. F. Kreimer; it simplifies an earlier proof due to the authors.
Thus $F(t)/F$ does not split and $F(t)^n$ is not simple by Theorem 2. Hence $(F(t)^n)_i$ is purely inseparable of exponent one and not simple. Thus $E/F$ is not simple.

To prove the converse we assume that $E/F$ is not simple and choose $a, b \in E$ such that $F(a, b)/F$ is not simple. Let $s \in K_* - F$ and, as in Lemma 3, set $t = a + bs$. Then $F(t)/F$ does not split and $F(a, b) = (F(t)^n)_i$. Moreover, $F(\psi) = F(s)$ is separable over $F$. Thus if $F(t) \cap F(a, b)$ properly contained $F$ then $F(t)/F$ would necessarily split. Hence $F(t)_i = F$ and $F(t)/F$ is exceptional.

**Corollary 7.** If $F(t)/F$ is inseparable but not purely inseparable and if $f = \sum_{i=0}^e a_i t^i$ is the minimum polynomial for $t$ over $F$, where $e = \exp f$, then $F(t)/F$ is exceptional if and only if $F(\{a_i^{1/p}\}_0)/F$ is not simple.

**Proof.** Sufficiency follows as in Theorem 2. Necessity follows from Theorem 6 and the fact that $F(\{a_i^{1/p}\}_0)$ is the maximal purely inseparable subfield of exponent one of $F(t)^n/F$.

In view of Theorem 6, if there exists a purely inseparable extension $L$ of $F$ such that $L/F$ is not simple and such that $E/F$ is simple where $E$ is the maximal subfield of $L/F$ of exponent one, then there exists a normal extension $K$ of $F$ such that $K/F$ is not simple, but there are no intermediate exceptional extensions. If we take $F = P(X, Y, Z)$ where $P$ is a perfect field and where $\{X, Y, Z\}$ is algebraically independent over $P$, and if $L = F(X^{1/p}, X^{1/p^2} + Y^{1/p}, X^{1/p^3}Z^{1/p})$, then it can be shown that $E = F(X^{1/p})$, providing the desired example.

**References**


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