AN INVARIANT FOR ALMOST-CLOSED MANIFOLDS

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1. Let $M^n$ be a compact, oriented, connected, $n$-dimensional differential manifold with $\partial M$ (boundary $M$) homeomorphic to the $n-1$ sphere $S^{n-1}$. Then $\partial M$ represents an element $[\partial M]$ of $\Gamma^{n-1}$, the group of differential structures (up to equivalence) on $S^{n-1}$. We consider the (much studied) problem of expressing $[\partial M]$ in terms of “computable” invariants of $M$.

Let $\pi_{n-1}$ be the $n-1$ stem, $J_0: \pi_n(\text{BSO}) \to \pi_{n-1}$ the classical $J$-homomorphism, and $\pi_{n-1}'$ the cokernel of $J_0$. In [5], a map $P: \Gamma^{n-1} \to \pi_{n-1}'$ was defined (see below). We will define an invariant $\Delta(M)$ which is a subset of $\pi_{n-1}'$ (and often consists of a single element). The main theorem states: $P[\partial M] \in \Delta(M)$.

In a strong sense, the definition of $\Delta(M)$ involves only homotopy theory. Moreover, $\Delta(M)$ seems amenable to computation by standard techniques of algebraic topology. We illustrate this below and, as applications, give explicit examples (1) of a manifold $M^n$, $n$ odd, with $[\partial M]^0$, and (2) of $M^n$, $n$ even, with $[\partial M]$ not only $\neq 0$, but in fact with $[\partial M]$ not even contained in $\Gamma^{n-1}(\partial \pi)$, the subgroup in $\Gamma^{n-1}$ of elements which bound $\pi$-manifolds. (Examples of $M^n$, $n$ even, with $[\partial M] \neq 0$ are of course well known.) Other applications, and detailed proofs, will appear elsewhere.

Remark 1. By [5], kernel $P = \Gamma^{n-1}(\partial \pi)$. If $n$ is odd, $\Gamma^{n-1}(\partial \pi) = 0$, so $P$ is injective, while if $n = 2$ (4), kernel $P \subseteq \mathbb{Z}_2$. If $n = 0$ (4), kernel $P$ tends to be large (but see §5).

Let $\text{BSO}$, $\text{BSPL}$, $\text{BSTop}$ be the stable classifying spaces for orientable vector bundles, piecewise-linear (= PL) bundles, topological bundles. There are maps $J_G: \pi_n(\text{BSG}) \to \pi_{n-1}$ ($G = \text{O}$, PL, Top) and a commutative diagram with exact rows

\[
0 \to \pi_n(\text{BSO}) \xrightarrow{f} \pi_n(\text{BSPL}) \xrightarrow{g} \Gamma^{n-1} \to 0
\]

\[
\pi_n(\text{BSO}) \xrightarrow{J_0} \pi_{n-1} \xrightarrow{q} \pi_{n-1}' \to 0.
\]

If $z \in \Gamma^{n-1}$, define $P(z)$ as $q(J_0(y))$, where $g(y) = z$.

2. On Thom complexes. Let $\beta$ be an oriented (topological) $k$-disk bundle over a CW-complex $X$, $T(\beta)$ the Thom complex. If $X$

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\[ Y \cup \delta e^n \quad (= Y \text{ with an } n\text{-cell attached by } \partial : \partial e^n \to Y), \] then \[ T(\beta) = T(\beta | Y) \cup \phi e^{n+1}. \] Also, if \( \ast \in Y \) is the basepoint, we have an inclusion \[ i : S^k = T(\beta | \ast) \to T(\beta | Y). \] Assume \( k, n \geq 2. \)

**Proposition 1.** Let \( X = Y \cup e^n, \) \( Y \) a connected \( n-1 \) dimensional complex. Let \( \alpha, \beta \) be oriented (topological) \( k \)-disk bundles over \( X \) with \( \alpha \mid Y \) isomorphic to \( \beta \mid Y. \) Let \( T(\beta) = T(\beta | Y) \cup e^{n+1}, \) \( [\phi] \in \pi_{n+k-1}(T(\beta | Y)). \) Suppose \( T(\alpha) \) is reducible \([4]\). Then \( [\phi] \in \text{image } i_\ast : \pi_{n+k-1}(S^k) \to \pi_{n+k-1}(T(\beta | Y)). \)

**Remainder 2.** If \( \delta \) is a \( k \)-disk bundle over \( S^n \) derived from \( (\alpha, \beta) \) by the difference construction, then in fact \( [\phi] = \pm i_* J(\delta), \) where \( J = J_{\text{top}} : \pi_n(\text{BSPL}) \to \pi_{n-1} = \pi_{n+k-1}(S^k) \) (here we assume \( k \) is large, although the remark has a nonstable analogue).

**Remainder 3.** Proposition 1 can be generalized to the case in which \( T(\alpha) \) is not necessarily reducible. One then has a statement about the difference of the attaching maps in the two Thom complexes.

3. **Definition of the invariant.** Given \( M^n \) as in \( \S 1, \) let \( M^* \) be the closed PL manifold \( M \cup \text{Cone}(\partial M). \) Let \( v_M \) be the \( k \)-dimensional normal bundle of \( M \) in Euclidean \( n+k \) space (\( k \) large). Using the fact that the map \( \pi_{n-1}(\text{BSO}) \to \pi_{n-1}(\text{BSPL}) \) is injective, one sees that \( v_M \) extends to a vector bundle \( v^* \) on \( M^*. \) Let \( T(v^*) = T(v_M) \cup e^{n+1}, \) \( [\phi] \in \pi_{n+k-1}(T(v_M)). \) Apply Proposition 1 with \( \alpha = v^\text{PL}(M^*) = k \)-dimensional PL normal bundle of \( M^*, \beta = v^*. \) We conclude that \( [\phi] \in \text{image } i_\ast : \pi_{n-1} = \pi_{n+k-1}(S^k) \to \pi_{n+k-1}(T(v_M)). \) Define \( \Delta'(v^*) \subseteq \pi_{n-1} \) as \( \{y \in \pi_{n-1} : i_\ast(y) = [\phi]\}. \) Let \( \Delta(v^*) = q(\Delta'(v^*)) \subseteq \pi_{n-1}. \) Now \( \Delta'(v^*) \) depends on the particular vector bundle extension \( v^* \) of \( v_M; \) \( \Delta(v^*), \) however, does not. We may therefore define:

\[ \Delta(M) = \Delta(v_M) = \Delta(v^*), \]

where \( v^* \) is any vector bundle on \( M^* \) extending \( v_M. \)

**Theorem 1.** Let \( M^n \) be a compact, oriented, connected, differential \( n \)-manifold with \( \partial M \) homeomorphic to \( S^{n-1}. \) Then \( \pm P[\partial M] \subseteq \Delta(M). \)

**Proof (Sketch).** Let \( v^\text{PL}(M^*), v^* \) be as above. It can be shown that there is a \( y \in \pi_n(\text{BSPL}) \) with \( g(y) = [\partial M] \) and such that \( y \) is a difference bundle for \( (v^\text{PL}(M^*), v^*). \) By Remark 2, \( T(v^*) = T(v_M) \cup e^{n+1}, \) where \( [\phi] = \pm i_* J(y). \) But \( q(J(y)) = P[\partial M]. \) Thus \( \pm P[\partial M] \subseteq \Delta(M). \)

4. **We give some applications of Theorem 1** (\( M \) is always as in \( \S 1). \)

**Definition.** A manifold \( M \) is of type \( m \) with respect to \( (X, \beta) \) if \( X \) is a CW-complex with \( m \) cells in positive dimensions, \( \beta \) is a vector
bundle over $X$, and there is a map $f: M \rightarrow X$ with $f^*(\beta)$ stably isomorphic to $\nu_M$.

We consider here manifolds of type one. This class of manifolds is certainly wide enough to be of geometric interest. For example, the following are of type one (with respect to $S^i$ and some $\beta \in \pi_i(\text{BSO})$).

(a) $i - 1$ connected $M^n$, $n = 2i$.
(b) $i - 1$ connected $M^n$, $n = 2i + 1$, $i \neq 1, 2$.
(c) The manifolds $M^n(g_1, g_2)$, where $g_1 \in \pi_{i-1}(\text{SO}(n-i))$ and $g_2 \in \pi_{n-i-1}(\text{SO}(i))$, formed by plumbing an $(n-i)$-disk bundle over $S^i$ (with characteristic map $g_1$) and an $i$-disk bundle over $S^n$ (with characteristic map $g_2$), provided that the bundle over $S^n_{i-1}$ is stably trivial.

Suppose $M^n$ is of type one with respect to $(S^i, \beta)$, and let $j: S^{n-1} \rightarrow M$ be the inclusion of $\partial M$ into $M$.

**Definition.** $\Phi_\beta(M) = \{ f \mid f: M \rightarrow S^i$ and $f^*(\beta)$ stably isomorphic to $\nu_M \}$ (Thus $\Phi_\beta(M) \subseteq \pi_{n-1}(S^i)$.)

$\Phi_\beta$ appears to be an important invariant for the study of manifolds of type one. We take the view that $\Phi_\beta(M)$ is "known" or computable. This is certainly reasonable for cases (a), (b), (c) above. For example, in cases (a) or (b) one can usually express $\Phi_\beta(M)$ in terms of more standard invariants (Pontryagin classes, behavior of cohomology operations, etc.) and in case (c) we have:

**Lemma 1.** Let $M^n = M(g_1, g_2)$ as in (c). Then $M$ is of type one with respect to $(S^i, g_1)$, and $J(g_2) \in \Phi_{g_1}(M)$. (Here $J: \pi_{n-i-1}(\text{SO}(i)) \rightarrow \pi_{n-1}(S^i)$.)

We wish to compute $\Delta(M)$ in terms of $\Phi_\beta(M)$.

**Theorem 2.** Let $M^n$ be of type one with respect to $(S^i, \beta)$, $\beta \in \pi_i(\text{BSO})$. Suppose the composition $xy \in \Phi_\beta(M)$, where $y \in \pi_{n-1}(S^p)$, $x \in \pi_p(S^i)$, $i < p < n - 1$; and suppose $x^*(\beta) = 0$. Then

(i) The Toda bracket $\langle J_0(\beta), S_p(x), S(y) \rangle$ is defined.
(ii) $\pm \Delta(M) \subseteq q \langle J_0(\beta), S_p(x), S(y) \rangle$.

**Explanation.** Here $S: \pi_{n-1}(S^p) \rightarrow \pi_{n-1-p}$ is the suspension map; $S_p: \{ x \in \pi_p(S^i) : x^*(\beta) = 0 \} \rightarrow \pi_{p-i}$ is a certain "twisted" suspension map, which we will not define here.

**Remark.** Theorem 2 can be generalized; for example, one may replace $S^p$ by an arbitrary complex.

**Lemma 2.** For a suitable generator $\gamma$ of $\pi_4(\text{BSO})$, $S_\gamma: \pi_7(S^4) \rightarrow \pi_3$ satisfies:
$S\gamma(H) = 0$, $H$ the Hopf map,

$S\gamma(t) = S(t)$, $t$ an element of finite order.

Recall that $\pi\theta(S^\delta) = Z_\delta \oplus Z_\delta = \{c\} \oplus \{d\}$, where (in notation of [9]) $c = Ev' \circ \eta\gamma, d = \nu\gamma \circ \eta\gamma$.

As an illustration of Theorem 2, we have

**Theorem 3.** Let $M^\gamma$ be of type one with respect to $(S^\delta, \gamma)$, $\gamma$ as in Lemma 2. Recall $\Gamma\gamma = Z_\delta$. Then

(i) If $O$ or $d \in \Phi\gamma(M)$, then $[\partial M] = 0$.

(ii) If $c$ or $c + d \in \Phi\gamma(M)$, then $[\partial M] \neq 0$.

**Proof (Sketch).** Suppose that $c \in \Phi\gamma(M)$. By Theorems 1 and 2, $P[\partial M] \in \Delta(M) \subseteq q\langle J(\gamma), S\gamma(Ev'), S(\eta\gamma) \rangle = q\langle J(\gamma), S(Ev'), S(\eta\gamma) \rangle$ (by Lemma 2). Using [9, especially Chapter VI], one calculates that this set is the nonzero element of $\pi\gamma'(Z_\delta)$. Other cases follow similarly.

**Examples.**

1. There is a $z \in \pi\gamma(SO(4))$ with $J(z) = c$. Consider the 9-manifold $M(g_1, g_2)$ with $g_1 \in \pi\gamma(SO(5))$ stably equal to $\gamma$ and $g_2 = z$.

2. There is a $w \in \pi\gamma(SO(4))$ with $J(w) = \alpha\gamma(4) \circ \alpha\gamma(7)$ [9, p. 178].

3. There is a $v \in \pi\gamma(SO(4))$ with $J(v) = Ev' \circ \varepsilon$ [9, p. 66].

**Remark.** Let $p$ be an odd prime, and let $n = 2p(p-1) - 1$. It is known that the $p$-primary component of $\Gamma\gamma = Z_\gamma$. Theorem 1 gives good information when applied to the problem of detecting the $p$-primary component of $[\partial M]$, $\dim M = n$. For example, one may show that if the $p$-primary component of $[\partial M] \neq 0$, then $g_1(M) \neq 0$, $g_1$ the first (mod $p$) Wu class. In particular, $M$ can not be $2(p-1)$-connected. It may be conjectured that the generator of the $p$-primary component of $\Gamma\gamma = Z_\gamma$ bounds a manifold of the homotopy type of $S^\gamma \vee S^{n-t}, t = 2(p-1)$. This is true if $p = 3$ (see Example 2 above).

5. The case $n = 4k$. Define $r: \Gamma\gamma \to Q/Z$ (rationals mod 1) as follows: given $z \in \Gamma\gamma$, choose $y \in \pi\gamma(BSPL)$ with $g(y) = z$. Then put $r(z) = (p_k(y))/b_k \bmod 1$, where $p_k$ is the $k$th rational Pontryagin class and $b_k = p_k(\gamma)$, $\gamma$ a generator of $\pi\gamma(BSO)$. (By Bott, $b_k = a_k(2k-1)!$, $a_k = 1$ ($k$ even) or 2 ($k$ odd).)
Define \( P' : \text{kernel } r \to \pi_{n-1} \) as follows: if \( r(z) = 0 \), there is a (unique) \( y \) with \( g(y) = z \) and \( p_k(y) = 0 \). Put \( P'(z) = J_{PL}(y) \).

**Lemma 3.** The pair \((r, P')\) is injective, in the sense that if \( r(z) = 0 \), then \( P'(z) \) is defined, and if \( P'(z) = 0 \), then \( z = 0 \).

**Proof.** Assume \( r(z) = 0 \), and let \( y \in \pi_n(BSPL) \) satisfy \( g(y) = z \), \( p_k(y) = 0 \). Then \( J_{PL}(y) = P'(z) = 0 \), by assumption. Thus \( p_k(y) = J_{PL}(y) = 0 \). But this implies \( y = 0 \) (see [2], [3], [8]), so \( z = 0 \).

Now given \( M^n \), define \( s(M) \) by

\[
s(M) = \left[ p_k(\nu^*) - p_k(\nu_{PL}(M^*)) \right]/b_k \mod 1,
\]

where \( \nu^* \) is any vector bundle on \( M^* \) extending \( \nu_M \).

If \( s(M) = 0 \), \( \nu^* \) may be chosen with \( p_k(\nu^*) = p_k(\nu_{PL}(M^*)) \). Then define \( \Delta'(M) \) as \( \{ x \in \pi_{n-1} : \iota_*(x) = [\phi] \} \), where \( T(\nu^*) = T(\nu_M) \cup_{\phi} e^{n+k} \) (as in §3).

**Theorem 1'.** (i) \( s(M) = r[\partial M] \). (ii) If \( s(M) = 0 \), then \( \pm P'[\partial M] \in \Delta'(M) \).

**Remark.** The invariant \( r \) is closely related to Milnor's \( \lambda \) invariant [7]. In fact, \( b_k \cdot r(z) = \lambda(z) \mod 1 \).

Let \( d_k \) be the denominator of \( B_k/4k \), \( B_k \) the \( k \)th Bernoulli number. Let \( j_k \) be the order of the image of \( J_0 : \pi_{4k}(BSO) \to \pi_{4k-1} \). Recall that \( j_k = t_k d_k \), \( t_k = 1 \) or 2. In every known case, \( t_k = 1 \) (for example, \( k \) odd [1], \( k = 2 \) or 4, or \( k \) as in [6]).

In the rest of this section, \( \dim M = 4k \), where \( t_k = 1 \).

**Theorem 4.** Let \( M \) be a spin manifold, and suppose \( \Delta(M) = 0 \). Then \( [\partial M] = 0 \) if and only if \( s(M) = 0 \) and \( \tilde{A}(M^*) \), the \( \tilde{A} \)-genus of \( M^* \), is integral.

**Proof.** Necessity is well known.

**Sufficiency.** Let \( T(\nu^*) = T(\nu_M) \cup_{\phi} e^{n+k} \), where \( p_k(\nu^*) = p_k(\nu_{PL}(M^*)) \). Let \( y \in \Delta'(M) \); i.e. let \( \iota_*(y) = [\phi] \). One may show that \( \tilde{A}(M^*) = e(y) \mod 1 \), where \( e \) is the invariant of [1]. Also, \( \Delta(M) = 0 \) implies \( y \in \text{image } J_0 \). But by [1], \( y \in \text{image } J_0 \) and \( e(y) = 0 \) imply \( y = 0 \) (if \( t_k = 1 \)). Thus \( \Delta'(M) = 0 \) and the theorem follows from Theorem 1' and Lemma 3.

**Example (Kervaire-Milnor).** Suppose \( \nu_M \) is the trivial bundle. Then \( [\Delta(M)] = 0 \) if and only if \( (p_k(M^*))/b_k \) and \( \tilde{A}(M^*) \) are integral.

**Proof.** One sees that \( \Delta(M) = 0 \) and that \( s(M) = (p_k(M^*))/b_k \mod 1 \). Apply Theorem 4.

\[ ^1 \text{We use a definition of } \tilde{A} \text{ which differs from the customary one by a factor of } 1/g_k. \]
6. Theorem 1 can be improved somewhat. Let $D(\nu_M)$ be the set of all differential structures on the topological manifold $M$ with normal bundle equal to $\nu_M$; by restricting each such structure to $\partial M$, we obtain a subset $\Gamma(\nu_M)$ of $\Gamma^n$. The argument in the proof of Theorem 1 shows that $\pm P(\Gamma(\nu_M)) \subseteq \Delta(\nu_M)$. (Using properties of the map $J_{PL}$ [2], [3], [8], one can sometimes show that this inclusion is an equality.)

REFERENCES