THE UNION OF FLAT \((n-1)\)-BALLS IS FLAT IN \(R^n\)

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THEOREM.\(^2\) Let \(\beta_1^{n-1}\) and \(\beta_2^{n-1}\) be two locally flat \((n-1)\)-balls in \(R^n\) with \(\beta_1 \cap \beta_2 = \partial \beta_1 \cap \partial \beta_2 = \beta^{n-2}\), where \(\beta^{n-2}\) is an \((n-2)\)-ball which is locally flat in \(\partial \beta_1\) and \(\partial \beta_2\). Then \(\beta_1 \cup \beta_2\) is a flat \((n-1)\)-ball in \(R^n\).

This result has been announced by Černavskii [1], but only for \(n \geq 5\) since his outlined proof uses engulfing. Our proof avoids engulfing and works for all \(n\); a thorough knowledge of Cantrell and Lacher’s version (see [2, §§4 and 5]) of Černavskii’s theorem is necessary to understand our proof.

We also have another proof of the following corollary which appears in [4].

COROLLARY. Let \(g: M^{n-1} \to N^n\) be an imbedding of an \((n-1)\)-manifold into an \(n\)-manifold which is locally flat except on a set \(E\). If \(n>3\), then \(E\) contains no isolated points (see [3] for the same result when \(M\) and \(N\) are spheres).

PROOF. Let \(C\) be a neighborhood of an isolated point \(p\) in \(M\) which is homeomorphic to an \((n-1)\)-ball, with \(g\) locally flat on \(C - p\). Then split \(C\) into \((n-1)\)-balls \(C_1\) and \(C_2\) so that \(C = C_1 \cup C_2\) and \(C_1 \cap C_2\) is an \((n-2)\)-ball containing \(p\). \(g\) is locally flat on \(C_1\) and \(C_2\) except at the point \(p\) on their boundaries. Then, since \(n>3\), \(g\) is flat on all of \(C_1\) and \(C_2\) by [5]. It follows from the theorem that \(C_1 \cup C_2 = C\) is flat, so \(E\) has no isolated points.

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2 Added in proof. Černavskii has independently proven this theorem by similar methods.
Let $R^n$ be Euclidean $n$-space, $B^n$ be the unit $n$-ball, and $R^k$ be imbedded in $R^n$ as $R^k = \{x \in R^n | x_{k+1} = \cdots = x_n = 0\}$. We will coordinatize $R^n$ by using $R^n = R^{n-2} \times R^2$ with polar coordinates on $R^2$. Thus points of $R^n$ will be triples $(z, r, \theta)$ with $z \in R^{n-2}$, $r \geq 0$, and $\theta \in R$ and with the convention that $(0, r, 0)$ is a point on the positive $x_n$-axis and $(r, \pi/2)$ is a point on the positive $x_n$-axis. Let $H_\phi = \{(z, r, \theta) \in R^n | \theta = \phi\}$ and $D_\phi = H_\phi \cap B^n$. Note that $D_\phi \cup D_0 = B^{n-1}$ and $D_\phi \cap D_0 = B^{n-2}$. Let $W(\theta_1, \theta_2)$ be the wedge $\{(z, r, \theta) | \theta_1 \leq \theta \leq \theta_2\}$ and $\tilde{W}(\theta_1, \theta_2) = W(\theta_1, \theta_2) \cap B^n$.

**Proof of Theorem.** Suppose $\beta_1$ and $\beta_2$ are given by imbeddings $\beta_1: D_\phi \to R^n$ and $\beta_2: D_0 \to R^n$. Since $\beta^{n-2}$ is locally flat in $\partial\beta_1$ and $\partial\beta_2$, the closures of $\partial\beta_1 - \beta^{n-2}$ and $\partial\beta_2 - \beta^{n-2}$ are homeomorphic to $(n-1)$-balls. Then we may assume that $\beta_1(D_\phi) \cap \beta_2(D_0) = \beta_1(B^{n-2}) = \beta_2(B^{n-2}) = \beta^{n-2}$.

Since locally flat imbeddings of balls are flat, $\beta_1$ and $\beta_2$ extend to imbeddings of $R^n$ into $R^n$ (still called $\beta_1$ and $\beta_2$). We can require that the extensions are chosen so that $\beta_1(H_\phi) \cap \beta_2(D_0) = \beta^{n-2}$ and $\beta_2(B^n) \subset \beta_1(R^n)$. Then it suffices to show that $D_\phi \cup \beta_1^{-1}\beta_2(D_0)$ is locally flat. Let $f = \beta_1^{-1}\beta_2$.

Since $f(D_0) \cap H_\phi = B^{n-2}$, we can assume that $f(D_0) \subset W(0, \pi/4)$ by rotating $f(D_0)$ around $R^{n-2}$ and away from $H_\phi$ while fixing $H_\phi$. Then, in the coordinates of $f(B^n)$, we can rotate $f(D_\phi)$ close to $f(D_0)$, so we may as well assume that $f(D_\phi) \subset W(0, \pi/4)$ and lies between $H_{\pi/4}$ and $f(D_0)$ (see Figure 1).

Let $h: R^n \to \text{int } H_0 \to \text{int } W(0, \pi/2)$ be the obvious homeomorphism which takes the wedge $W(0, \pi) - \text{int } H_0$ onto $W(\pi/2, \pi) - \text{int } H_{\pi/2}$ and fixes int $W(\pi, 2\pi)$. The set $W(0, \pi) \cap f(B^n)$ is separated.

**Figure 1**

**Figure 2**
into two sets by \(f(D_{\pi/2})\); let \(T\) denote the set containing \(f(D_0)\). Then (see Figure 2) define an imbedding \(h: f(B^n - D_{\pi/2} \cup B^{n-2}) \to \mathbb{R}^n\) by

\[
h(f(x)) = f(x) \quad \text{if } f(x) \in T,
\]

\[
= \bar{h}(x) \quad \text{if } f(x) \notin T.
\]

To ensure that \(h\) is an imbedding it may be necessary to trim away part of \(f(B^n)\), still leaving a “ball-neighborhood” of \(f(D_0)\) (in Figure 3, restricting to the dotted ball would eliminate the annoying feelers). Note that \(hf=f\) on \(D_0\) and \(hf(D_{\pi/2}) \subset W(\pi/2, \pi)\).

We need to extend \(hf|\overline{W}(\pi, 2\pi)\) to an imbedding of \(B^n\) into \(\mathbb{R}^n\). We can assume that for some \(\varepsilon>0\), \(f(D_{2\pi-\varepsilon}) \subset W(0, \pi/2)\), so then \(h=f\) on \(D_{2\pi-\varepsilon}\). Let \(g_1\) be the homeomorphism of \(B^n - D_{\pi/2} \cup B^{n-2}\) which fixes points outside \(\overline{W}(3\pi/4, 2\pi)\) and moves \(D_{2\pi-\varepsilon}\) to \(D_{\pi}\). Let \(g_2: hf(\overline{W}(3\pi/4, 2\pi-\varepsilon)) \to hf(\overline{W}(3\pi/4, \pi))\) be the homeomorphism defined by \(g_2 = hf g_1(hf)^{-1}\). Now define an imbedding \(g: f(\overline{W}(0, 2\pi-\varepsilon)) \to \mathbb{R}^n\) by

\[
g(x) = g_2(x) \quad \text{if } x \in hf(\overline{W}(3\pi/4, 2\pi-\varepsilon)),
\]

\[= x \quad \text{otherwise}.
\]

To make sure that \(g\) is well defined, it may be necessary to again shrink \(f(B^n)\) towards \(B^{n-2}\) so that \(\text{int } f(\overline{W}(0, 2\pi-\varepsilon)) \cap \partial hf(\overline{W}(3\pi/4, 2\pi-\varepsilon)) \subset hf(D_{2\pi/4})\). Let \(i: \overline{W}(0, \pi) \to \overline{W}(0, 2\pi-\varepsilon)\) and note that \(gf = hf\) on \(D_\pi\).

Then (see Figure 4), we can piece together \(gf\) and \(hf\) to get an imbedding \(F: B^n \to \mathbb{R}^n\); specifically, let

\[
F(x) = gf(x) \quad \text{if } x \in \overline{W}(0, \pi),
\]

\[
= hf(x) \quad \text{if } x \in \overline{W}(\pi, 2\pi).
\]

\(F = f\) on \(D_0\), so \(F(D_0) \subset W(-\pi/2, \pi/2)\), and \(F(D_{\pi}) = hf(D_{\pi})\).
Thus $F(B^{n-1})$ is “transverse” to $H_{\pi/3} \cup H_{3\pi/3}$, and that is the key to the proof. It allows us to find an isotopy making $F(D_\theta)$ tangent to $H_\theta$ at $B^{n-2}$ for all $\theta$. This isotopy is constructed in the latter part of the proof of Lemma 5.2 of [2]. Then a homeomorphism of $R^n$ can be constructed which fixes $D_x$ and takes $F(D_0)$ to $D_0$ (see the proof of Theorem 6.1 in [2]). Thus $(R^n, \beta_1 \cup \beta_2)$ is pairwise homeomorphic to $(R^n, D_x \cup D_0)$, finishing the proof.

References


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