SOME THEOREMS ON FACTORIZATION OF MEROMORPHIC FUNCTIONS

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In [3] the author proved

**Theorem 1.** If \( f \) is any entire function of lower order less than \( \frac{1}{2} \) and \( g \) is entire, then \( f(g) \) is periodic if and only if \( g \) is.

By means of a result due to Edrei [1] and Ostrovskii [6] it is possible to generalize Theorem 1 to a certain class of meromorphic functions. We begin with

**Lemma 1 (Edrei [1], Ostrovskii [6]).** Let \( f(z) \) be meromorphic of lower order \( \lambda < \frac{1}{2} \). If \( \delta(\infty, f) > 1 - \cos \pi \lambda \), then \( |f(z)| \to \infty \), uniformly in \( \theta \) as \( r_n \to \infty \) through a suitable sequence.

Here \( \delta \) is the Nevanlinna deficiency (see Hayman [5, p. 42]).

**Theorem 2.** Let \( f \) be meromorphic of lower order \( \lambda \) and let \( g \) be entire. If \( 0 \leq \lambda < \frac{1}{2} \) and for some \( a, \delta(a, f) > 1 - \cos \pi \lambda \), then \( f(g) \) is periodic if and only if \( g \) is. If \( \tau \) is a period of \( f(g) \), then \( g \) has a period having the same argument as \( \tau \).

**Sketch of Proof.** We assume that \( f(g) \) is periodic with period \( \tau \) having argument \( \alpha \). Let \( L \) be the half line \( re^{i\alpha} \) everywhere except near poles of \( f(g) \), where we let \( L \) loop around them with radius \( \epsilon, \epsilon \) a small positive number. Letting \( f^*(z) = 1/(f(z) - a) \) and applying Lemma 1 we see that \( |f^*(re^{i\theta})| \to \infty \), uniformly in \( \theta \) as \( r_n \to \infty \) through a suitable sequence. From the hypotheses of the theorem it follows that \( f(g) \) is bounded on \( L \). If \( g \) is bounded on \( L \), then as in the proof of Theorem 1 (see [3]) \( g \) must be periodic with a period having the same argument as \( \tau \). If \( g \) is unbounded on \( L \), then \( f \) is bounded on \( g(L) \) and this leads to a contradiction via Lemma 1.

**Corollary.** If \( P \) is a polynomial and \( f \) is as in Theorem 2, then \( f(P) \) is not periodic.

This Corollary is a partial solution to the more general question suggested in [4]: If \( f \) is meromorphic for which polynomials is \( f(P) \) periodic?

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Theorem 2 also yields a generalization of an earlier result mentioned in [4].

**Theorem 3.** Let $f$ be meromorphic and $g$ entire. If $f(g)$ is of finite order, has no deficient values and is periodic, with period $\tau$, then either $f$ has no deficient values or $g$ is periodic with a period having the same argument as $\tau$.

**Sketch of Proof.** By a theorem of Edrei and Fuchs [2] either $f$ is of zero order or $g$ is a polynomial. In the latter case $f$ can certainly not have any deficient values since $f(g)$ does not. In the former case one can apply Theorem 2 and arrive at the desired conclusion.

**Corollary (See [4]).** Let $f$ be meromorphic and $g$ entire. If $f(g)$ is elliptic, then $f$ has no deficient values.

This last corollary rules out the possibility of applying the earlier one to resolve the question mentioned in [4]: If $P$ is a polynomial of degree $n$, where $n = 5$ or $n \geq 7$ and $f$ is any meromorphic function, then $f(g)$ is not elliptic?

**References**

2. A. Edrei and W. H. Fuchs, *On the zeros of $f(g(z))$ where $f$ and $g$ are entire functions*, J. Analyse Math. 12 (1964), 243–255.

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