ON THE SYMBOL OF A PSEUDO-DIFFERENTIAL OPERATOR

BY WEISHU SHIH

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In [1] Hörmander defines the generalized symbol of a pseudo-differential operator \( P \) as a sequence of partially defined maps between function spaces. Our purpose here is to comment on the existence of characteristic polynomial type symbols \( \sigma(P) \) and to obtain their composition by introducing a product structure on suitable jet bundles. In particular, this gives the lower order symbol for differential operator on manifold. I express my hearty thanks to J. Bokobza, H. Levine, and A. Unterberger for their indispensable help.

1. Operation in jet bundle. Given a compact \( C^\infty \) differentiable manifold \( X \), we denote by

\[
p_i : J^m(R) \to J^k(R), \quad m \geq k,
\]

the jet bundle of the trivial bundle \( X \times \mathbb{R} \) and the canonical projection. Identify the cotangent bundle \( T(X) \) as a subbundle of \( J^1(R) \) we define the subbundle

\[
J^k_0(R) \subseteq J^k(R), \quad k \geq 1,
\]
as the inverse image by \( p_i : J^k(R) \to J^1(R) \) of the nonzero cotangent vector \( T_0(X) \subseteq T(X) \). Let \( E, F, \) and \( G \) be complex vector bundles over \( X \) and put

\[
J^*(E, F) = \prod_{k=0}^\infty \text{Hom}(J^{k+1}_0(R) \oplus J^k(E), F)
\]
where "Hom" denotes the space of \( C^\infty \) bundle maps which are linear with respect to \( J^k(E) \). We shall construct an operation

\[
o : J^*(E, F) \times J^*(F, G) \to J^*(E, G)
\]
as follows. If \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots) \in J^*(E, F), \beta = (\beta_0, \beta_1, \ldots) \in J^*(F, G) \), then

\[
\alpha \circ \beta = (\gamma_0, \gamma_1, \ldots, \gamma_r, \ldots) \in J^*(E, G)
\]
is given by

\[
\gamma_r = \sum_{m+n=r} \beta_n \circ (p_{n+1} \circ p_R \oplus j^n(\alpha_m))
\]

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where \( p_R : J^*_0(R) \oplus J^*(E) \to J^*(R) \) is the projection, and

\[
j^n : \text{Hom}(J^{k+1}_0(R) \oplus J^k(E), F) \to \text{Hom}(J^{k+n+1}_0(R) \oplus J^{k+n}(E), J^n(F))
\]

the \( n \)th jet extension map.

**Theorem.** The operation \( \circ \) is well defined, associative, and distributive. Moreover if the \( \alpha_m \) (resp. \( \beta_n \)) in \( \alpha \) (resp. \( \beta \)) is positive homogeneous of degree \( k-m \) (resp. \( h-n \)) with respect to \( J^{m+1}_0(R) \), then \((\alpha \circ \beta)_r \) is positive homogeneous of degree \( k+h-r \) (\( k, h \) real numbers). In particular, with respect to this operation \( J^*(E, E) \) becomes an associative algebra with unity [2].

2. **The symbol homomorphism.** Let us recall that a continuous linear map

\[
P : C^\infty(E) \to C^\infty(F)
\]

between the space of \( C^\infty \) sections of complex vector bundles is a pseudo-differential operator of order \( k \) in the sense of Hörmander if:

for each \( g \in C^\infty(E), g \in C^\infty(R) \), such that

\[\text{(*) } \text{supp } f \subseteq \text{supp } dg \text{; interior of support of } dg\]

there is a uniform asymptotic expansion [1]

\[
e^{-\lambda \partial} P(e^{\lambda \partial} f) \sim \sum_0^\infty P_j(g, f) \lambda^{k-j}
\]

with \( P_j(g, f) \in C^\infty(F) \) and \( P_0(g, f) \neq 0 \). The formal sum \( \sum_0^\infty P_j(g, f) \) is the generalized symbol of \( P \).

Now let us denote by

\[
\varphi(E, F) = \sum_k \varphi_k(E, F)
\]

the space of all pseudo-differential operators from the complex vector bundle \( E \) to the bundle \( F \) over the fixed compact manifold \( X \); \( \varphi_k(E, F) \) those of order \( k \). Then we have

**Theorem.** There exists a unique homomorphism

\[
\sigma : \varphi(E, F) \to J^*(E, F)
\]

satisfying the following conditions:

1. If \( P \in \varphi_k(E, F) \), then \( \sigma_j(P) \) is positive homogeneous of degree \( k-j \) with respect to \( J^{m+1}_0(R) \) where \( \sigma(P) = (\sigma_0(P), \sigma_1(P), \cdots) \).
2. If \( P \in \varphi(E, F), Q \in \varphi(F, G) \), then

\[
\sigma(P \circ Q) = \sigma(P) \circ \sigma(Q).
\]
(3) If $P \in \mathcal{D}_k(E, F)$ and $f \in C^\infty(E)$, $g \in C^\infty(R)$, verify the condition (*), then the generalized symbol of Hörmander $P_j(g, f)$ is equal to the image of $(dg, f)$ by the composition

$$C^\infty(T_0(X)) \times C^\infty(E) \to C^\infty(J_0^{j+1}(R) \oplus J^j(E)) \xrightarrow{\sigma_j(P)} C^\infty(F)$$

restricted on the interior of the support of $dg$.

(4) If $P$ is a $k$th order differentiable operator, then $\sigma_j(P)$ is defined on $J^{j+1}(R) \oplus J^j(E)$ and is zero for $j > k$. Moreover the restriction of $\sigma_0(P)$ on $T_0(X) \subseteq J^1(R)$:

$$\sigma_0(P) : T_0(X) \oplus E \to F$$

is the classical [3] symbol of the differential operator $P$.

**Remark:** $\sigma : \mathcal{D}(E, E) \to J^*(E, E)$ is a homomorphism of algebra with unity. Choose a splitting (e.g. by connections). We obtain an inclusion $T_0(X) \oplus E \hookrightarrow J_0^{j+1}(R) \oplus J^j(E)$; then the restriction of $\sigma_j(P)$ on $T_0(X) \oplus E$ gives the lower order characteristic polynomial of $P$ (e.g. in the case of $\mathbb{R}^n$ one gets back the ordinary total characteristic polynomial of a differential operator). Using jet bundles [5] along the fiber, one obtains the same result for a family of operators.

**References**


Institute for Advanced Study and Institute Hautes Études Scientifiques