Let $L_p$ be a Banach function space, i.e. a Banach space of (equivalence classes of) measurable point functions on a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, with $\rho$ being a function norm possessing at least the weak Fatou property. The results obtained concern integral representations of bounded linear operators from a Banach space $\mathcal{X}$ to $L_p$ and from $L_p$ (or a subspace) to $\mathcal{X}$. These results in some cases complement and in other cases generalize work done in [1], [3], [5], [6], [7], [12], [13].

General notation and results on Banach function spaces can for the most part be found in the first parts of [11]; more detailed work is in [9].

A few further definitions are needed here. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, let $B(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Distinguish two subrings of $\Sigma$ as $\Sigma_0 = \{ E \in \Sigma : \rho(\chi_E) < \infty \}$ and $\Sigma' = \{ E \in \Sigma : \rho'(\chi_E) < \infty \}$. A partition $\mathcal{E}$ is defined to be a finite disjoint collection of non-$\mu$-null members of $\Sigma_0$ which are of finite measure. The "averaged" step function of a member $f$ of $L_p$ is defined as

$$ f_{\mathcal{E}} = \sum_{\mathcal{E}} \left( \int_{E_i} |f| \, d\mu / \mu(E_i) \right) \chi_{E_i}. $$

A function norm $\rho$ is said to have property (J) if, for each partition $\mathcal{E}$, $\rho(f_{\mathcal{E}}) \leq \rho(f)$. (This is very similar to the levelling property of [5].)

1. The structure of the space $B(\mathcal{X}, L_p)$.

   DEFINITION 1. We define a space of set functions: $\mathcal{U}_p = \{ x^*(\cdot) \mid x^*(\cdot) : \Sigma_0' \to \mathcal{X}^*, x^*(\cdot)\chi \text{ is countably additive and } \mu\text{-continuous for each } \chi \in \mathcal{X}, \text{ and } V_p(x^*(\cdot)) < \infty \}$ where

$$ V_p(x^*(\cdot)) = \sup_{\|x\| \leq 1} \sup_{\mathcal{E}} \rho \left( \sum_{\mathcal{E}} \frac{x^*(E_i) \chi}{\mu(E_i)} \chi_{E_i} \right). $$

The representation of bounded linear operators from $\mathcal{X}$ to $L_p$ is made in terms of this space.

---

1 The results announced here are contained in the author's doctoral dissertation written at Carnegie Institute of Technology under the guidance of Professor M. M. Rao.
Theorem 1. If \( p' \) possesses property (I), then there is an isomorphism between \( B(\mathfrak{X}, L_p) \) and \( \mathfrak{V}_p \); moreover,
\[
\gamma\|T\| \leq V_p(x^*(\cdot)) \leq \gamma^{-1}\|T\|
\]
for corresponding elements.

(The constant \( \gamma \) is fixed for each \( L_p \) space and has value \( 0 < \gamma \leq 1 \) with \( \gamma = 1 \) if and only if \( p \) has the strong Fatou property. It is the constant which appears in Theorems 1.1 and 1.2 of [11].)

The correspondence is obtained in one direction by defining \( x^*(E)x = \int_E Tx(\omega)d\mu(\omega) \) for \( E \in \Sigma_0' \). In the other direction, a type of Radon-Nikodym derivative is used.

2. The structure of the space \( B(M_p, \mathfrak{X}) \). Results for \( B(L_p, \mathfrak{X}) \) unfortunately seem not to be, in general, available by the present techniques. Results which will be presented in §3 have been obtained for the linear functionals (\( \mathfrak{X} = \text{scalars} \)). In the case of a general \( \mathfrak{X} \), we have results for a closed subspace of \( L_p \) (which in some common cases is all of \( L_p \)).

Definition 2. Let \( M_p = \text{cl}\{f \in L_p: f \text{ is bounded and has support in } \Sigma_0\} \).

Note that \( M_p \) is a closed subspace which is normal and a sublattice (in fact, a lattice ideal).

Definition 3. We define a space of set functions:
\[
\mathcal{W}_p' = \{x(\cdot)|x(\cdot): \Sigma_0 \rightarrow \mathfrak{X}, x(\cdot) \text{ is finitely additive, vanishes on } \mu\text{-null sets, and } W_p'(x(\cdot)) < \infty \},
\]
where
\[
W_p'(x(\cdot)) = \sup_{\|x\|_1 < 1} \sup_{s} \rho' \left( \sum_{i} \frac{x^*(E_i)}{\mu(E_i)} x_{E_i} \right).
\]

To represent elements of \( B(M_p, \mathfrak{X}) \) it will be desirable to integrate members of \( M_p \) against set functions in \( \mathcal{W}_p' \). In order to do this, Bartle's treatment [2] of integration will be used.

Definition 4. A measurable function \( f \) is integrable over \( \Omega \) with respect to an \( \mathfrak{X} \)-valued finitely additive set function \( x(\cdot) \) if there is a sequence \( \{f_n\} \) of simple functions such that
(i) \( f_n \rightarrow f \) in \( x(\cdot) \) measure,
(ii) \( \lambda_n(\cdot) \) are uniformly absolutely continuous, and
(iii) \( \lambda_n(\cdot) \) are equicontinuous,
where \( \lambda_n(E) = \int_E f_n dx \) for \( E \in \Sigma \) and where (i), (ii), and (iii) are with respect to the semivariation of \( x(\cdot) \).

The fact that if \( f \in M_p \) and \( x(\cdot) \in \mathcal{W}_p' \) then \( f \) is \( x(\cdot) \) integrable leads to the representation theorem:
Theorem 2. If \( p' \) has (J), then \( B(\mathcal{M}_p, \mathcal{N}) \) and \( \mathcal{N}_p' \) are isomorphic; moreover, \( \|T\| \leq W_p'(x(\cdot)) \leq \gamma^{-1}\|T\| \) for corresponding elements.

The correspondence is given by \( Tf = \int fdx \) and \( x(E) = T(x_E) \) for \( E \in \Sigma_0 \).

Corollary. If every member of \( \mathcal{M}_p \) has absolutely continuous norm, then each \( x(\cdot) \in \mathcal{N}_p' \) is \( \mu \)-continuous.

3. Representation of linear functionals. We have obtained two characterizations of \( L^* \). One assumes property (J), the other does not. Both results proceed by use of the quotient space \( (L_p/M_p)^* \).

Define \( N_p = L_p/M_p \) and equip \( N_p \) with the usual factor norm and order (recalling that \( M_p \) is a lattice ideal). Denote the canonical map as \( \lambda: L_p \to N_p \). Note that \( \lambda \) is continuous, interior, homomorphic (both linear and lattice), and has norm \( \leq 1 \). In addition \( \lambda^*: N_p \to M_p \) is an isometric isomorphic surjection. (Note that \( N_p \) is an \( AB \) lattice, even though it is not a Banach function space over the given measure space.) In \( L_p \) we define the convex, norm-determining and (in general) nonlinear subset \( L_p = \{ f: f = \sqrt[n]{f_i}, f_i \geq 0, \rho(f_i) \leq 1, 1 \leq n < \infty \} \). Since \( \lambda \) is interior, \( \lambda(L_p) \) contains the nonnegative elements of the open unit ball of \( N_p \). It is here that an assumption is needed:

Condition (I). \( \lambda(L_p) \) lies in the closed unit ball of \( N_p \). (With this assumption, \( N_p \) is an AL space in the sense of Kakutani.)

The characterization of \( N_p^* \) is in terms of certain additive set functions. Define \( \text{ba}(\Omega, \Sigma, \mu) \) to be the collection of bounded additive set functions on \( \Sigma \) which vanish on \( \mu \)-null sets. Denote by \( \text{ca}(\Omega, \Sigma, \mu) \) the countably additive members of \( \text{ba}(\Omega, \Sigma, \mu) \) and by \( \text{pfa}(\Omega, \Sigma, \mu) \) the purely finitely additive members of \( \text{ba}(\Omega, \Sigma, \mu) \). We will need to integrate elements of \( N_p \) with respect to set functions in \( \text{pfa}(\Omega, \Sigma, \mu) \).

The integration used is a variant of that found in [14] and [15].

Definition 5. Let \( 0 \leq \nu \in \text{pfa}(\Omega, \Sigma, \mu) \) and \( 0 \leq f \in L_p \). Define \( I_{\nu}(f) = \inf \{ \sum_{i=1}^{n} ||\lambda(f_E_i)||_{\nu(E_i)} : \{ E_i \} \text{ disjoint finite partition of } \Omega \} \).

This has all the usually desired properties of an integral and is extended to all of \( L_p \) and \( \text{pfa}(\Omega, \Sigma, \mu) \) by linearity on the decomposition into their positive parts. Note that one could equally well write \( I_{\nu}(\lambda(f)) \) since \( I_{\nu} \) is constant over cosets.

Theorem 3. Assuming that condition (I) holds, there is an isometric isomorphism which is also a lattice isomorphism between \( N_p^* \) and a closed subspace of \( \text{pfa}(\Omega, \Sigma, \mu) \) which shall be denoted as \( \Phi_p \). The isometry is \( ||\xi^*|| = ||\nu||(\Omega) \).

The correspondence is given by: for \( \nu \in \Phi_p, \xi^*(\lambda(f)) = I_{\nu}(f), f \in L_p; \) for \( 0 \leq \xi^* \in N_p^*, \nu(E) = ||\xi^*_E||, E \in \Sigma; \) and for general \( \xi^* \in N_p^*, \) one
uses its decomposition into positive parts. (We denote $s_f^*(\lambda(f)) = s^*(\lambda(f)\chi_E).$)

The space $p_f$ is determined as the range of the (bounded) projection obtained by composing the correspondence from $\text{pfa}(\Omega, \Sigma, \mu)$ into $N$ with that from $N_p^*$ to $\text{pfa}(\Omega, \Sigma, \mu)$. One may describe $p_f$ as those elements in $\text{pfa}(\Omega, \Sigma, \mu)$ whose support lies inside the support of a function in $L_p$ which is not in $M_p^*$.

**Theorem 4.** The conjugate space $L_p^*$ has a direct sum decomposition into two closed linear (lattice) subspaces which are seminormal, namely into $M_p^*$ and its lattice orthogonal complement $(M_p^*)^\perp$ which is isometrically linearly and lattice isomorphic to $M_p^*$. Moreover, in the decomposition, $\|x^*\| = \|y^*\| + \|z^*\|$ where $x^* = y^* + z^*$ with $y^* \in M_p^*$ and $z^* \in (M_p^*)^\perp$.

Thus in order to represent $L_p^*$ all that is needed is a representation of $M_p^*$. If $p'$ has property (J), then by Theorem 2, $M_p^*$ is isomorphic to the space of set functions $\mathcal{W}' = \{G | G: \Sigma_0 \to \text{reals}, G \text{ finitely additive on } \Sigma_0, G \text{ vanishes on } \mu\text{-null sets, and } \mathcal{W}_p(G) = \sup_G p'\left(\sum_G(E_i)/\mu(E_i)\chi_E\right) < \infty\}$ under the correspondence $G(E) = x^*(\chi_E)$ for $E \in \Sigma_0$, and $x^*(f) = \int f dG$ for $f \in M_p$. For corresponding elements, one has $\|x^*\| \leq W_p(G) \leq \gamma^{-1}\|x^*\|$. (It is also true that this correspondence is a lattice isomorphism.)

If we define $\alpha_p = \mathcal{W}_p^* \times p_f$ with norm $\|(G, \psi)\| = W_p(G) + \|\psi\|\Omega$ and with partial order $(G, \psi) \succeq (0, 0)$ if $G \geq 0$ and $\psi \geq 0$, then $\alpha_p$ is a Banach lattice and we have:

**Theorem 5.** If condition (I) holds and $p'$ has property (J), then the space $L_p^*$ is linear and lattice isomorphic to $\alpha_p$. Moreover the correspondence is a topological equivalence.

There is another characterization of $M_p^*$ that is available without the assumption that $p'$ has (J). However, this form is not as useful since the norm computation does not explicitly involve the associate space (although if $p'$ has (J), this approach leads to a norm equivalent to the one given above).

**Theorem 6.** There is an isometric isomorphism between $M_p^*$ and the Banach space $\mathcal{U} = \{\nu | \nu(\cdot) \text{ real valued, additive set function on } \Sigma_0 \text{ which vanishes on } \mu\text{-null sets, and } \|\nu\| = \sup\{\|f d\nu\| : f \text{ simple and } p(f) \leq 1\} < \infty\}$. Moreover, the members of $\mathcal{U}$ are all countably additive if and only if every function in $M_p$ is of absolutely continuous norm.

If we define $\alpha_p$ as $\mathcal{U} \times p_f$ with norm $\|(\nu, \psi)\| = \|\nu\| + \|\psi\|\Omega$ and
partial order \((v, \psi) \geq (0, 0)\) if \(v \geq 0\) and \(\psi \geq 0\), then \(\mathfrak{B}_v\) is a Banach lattice and we have:

**Theorem 7.** Under the assumption of (I) alone, the space \(L_p^*\) is linearly and lattice isomorphic to \(\mathfrak{B}_v^*\). Moreover the correspondence is an isometry.


**References**


Carnegie-Mellon University and University of California, Riverside

*Notes XIV, XV, and XVI are by W. A. J. Luxemburg alone.*