PROOF OF A CONJECTURE OF HELSON

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Let $m_n$ denote the Haar measure of the torus $T^n$, the distinguished boundary of the unit polydisc $U^n$ in the space of $n$ complex variables. If $f$ is holomorphic in $U^n$, define

\[ f^*(z) = \lim_{r \to 1} f(rz) \]

for those $z \in T^n$ for which this radial limit exists. Here $z = (z_1, \ldots, z_n)$, $rz = (rz_1, \ldots, rz_n)$. The various $H^p$-norms in $U^n$, for $0 < p < \infty$, $n = 1, 2, 3, \ldots$, are defined by

\[ \|f\|_{p,n} = \sup_{0 < r < 1} \left\{ \int_{T^n} |f(rz)|^p dm_n(z) \right\}^{1/p}. \]

As in one variable, the inequality

\[ \log |f(0)| \leq \int_{T^n} \log |f^*(z)| \, dm_n(z) \]

holds for every $f \in H^p(U^n)$. Define

\[ \Delta(f) = \int_{T^n} \log |f^*(z)| \, dm_n(z) - \log |f(0)|. \]

For $f \in H^2(U^n)$, let $S(f)$ denote the $H^2$-closure of the set of all products $Pf$, where $P$ ranges over the polynomials in $n$ variables; $S(f)$ is the invariant subspace of $H^2(U^n)$ generated by $f$.

A very well-known theorem of Beurling states (in one variable) that

\[ S(f) = H^2(U) \text{ if and only if } \Delta(f) = 0. \]

One of these implications holds equally well for several variables, as has been known for quite some time to Helson and Lowdenslager: If $f \in H^2(U^n)$ and $S(f) = H^2(U^n)$, then $\Delta(f) = 0$. Here is a sketch of a simple proof: (i) $\Delta(Pf) = \Delta(P) + \Delta(f) \geq \Delta(f)$ for all $P$. (ii) $\Delta$ is an upper semicontinuous function on $H^2(U^n)$. (iii) Therefore $\Delta(g) \geq \Delta(f)$ for every $g \in S(f)$.

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Helson has conjectured that the converse is false for $n = 2$ (hence also for $n > 2$). Actually, Helson stated the problem somewhat differently, in terms that involve only the boundary values of the functions under consideration. This conjecture is correct:

**Theorem.** There exists a function $f \in H^2(U^2)$ such that $\Delta(f) = 0$ but $S(f) \not\in H^2(U^2)$.

The proof depends on the following two observations.

1. If $F \in H^\infty(U)$, if $F$ has no zero in $U$, and if $f \in H^\infty(U^2)$ is defined by
   \begin{equation}
   f(z_1, z_2) = F((z_1 + z_2)/2),
   \end{equation}
   then $\Delta(f) = 0$.

2. Associate to each $f \in H^2(U^2)$ the function
   \begin{equation}
   \Psi f(\lambda) = f((1 + \lambda)/2, (1 + \lambda)/2) \quad (\lambda \in U).
   \end{equation}
   If $0 < p < \frac{1}{2}$, there is a constant $C_p < \infty$ such that
   \begin{equation}
   \|\Psi f\|_{p, 1} \leq C_p \|f\|_{2, 2}.
   \end{equation}
   Thus $\Psi$ maps $H^2(U^2)$ into $H^p(U)$ if $p < \frac{1}{2}$. Note that $\Psi f$ is essentially the restriction of $f$ to a certain disc in $U^2$ which touches $T^2$ at just one point.

**Proof of (I).** If $|\alpha| = 1$, $z \mapsto \alpha z$ is a measure-preserving map of $T^2$ onto $T^2$. Hence

\begin{equation}
\int_{T^2} dm_2(z) \int_T \log |f^*(\alpha z)| \ dm_1(\alpha) = \int_{T^2} \log |f^*(z)| \ dm_2(z),
\end{equation}

as is seen by interchanging the integrations on the left. If $z = (z_1, z_2) \in T^2$, if $z_1 \neq z_2$, and if $2w = z_1 + z_2$, then $|w| < 1$, so that

\begin{equation}
\log | F(0)| = \int_T \log | F(\alpha w) | \ dm_1(\alpha).
\end{equation}

This says that the inner integral on the left of (9) is equal to $\log |f(0)|$ whenever $z_1 \neq z_2$, which is true for almost all $z \in T^2$. Hence $\Delta(f) = 0$.

**Proof of (II).** For simplicity, assume $\|f\|_{2, 2} = 1$. Apply the Schwarz inequality to the Cauchy formula

\begin{equation}
f(\xi, \eta) = \int_{T^2} \frac{f^*(z_1, z_2)}{(1 - \bar{z}_1 \xi)(1 - \bar{z}_2 \eta)} \ dm_2(z)
\end{equation}

to obtain the estimate
\[ |f(\xi, \zeta)| \leq \left\{ \int_{\mathbb{T}^2} |1 - \overline{\xi} \zeta|^{-2} |1 - \overline{\xi_2} \zeta_2|^{-2} dm_2(z) \right\}^{1/2} \]

\[ = \int_{\mathbb{T}} |1 - \overline{w} \xi|^{-2} dm_1(w) = (1 - |\zeta|^2)^{-1} \]

if \(|\xi| < 1\). For \(\lambda = re^{i\theta}, 0 < r < 1\), it follows that

\[ |(\Psi f)(\lambda)| \leq \left\{ 1 - \frac{1 + \lambda}{2} \right\}^{-1} \leq \{ r \sin^2 (\theta/2) \}^{-1} \]

which gives (8) with

\[ C_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin(\theta/2)|^{2p} d\theta \right\}^{1/p}. \]

**Proof of the Theorem.** Put \(F(\lambda) = \exp \{ (\lambda + 1)/(\lambda - 1) \}\) and associate \(f\) with \(F\) as in (I). Then \(\Delta(f) = 0\).

Fix \(q, 0 < q < \frac{1}{2}\). If \(P\) is any polynomial in two variables, (II) gives

\[ \|1 - Pf\|_{2,2} \geq C_p^{-1}\|1 - \Psi P \cdot \Psi f\|_{p,1}. \]

Note that \((\Psi f)(\lambda) = e^{-1}F^2(\lambda)\). Thus \(e\Psi f\) is a nontrivial inner function in \(U\). Since multiplication by an inner function is an isometry in \(H^p(U)\) (relative to the metric given by \(\|g - h\|_{p,1}\) if \(p < 1\)) one sees that \(H^p(U)\Psi f\) is a closed subspace of \(H^p(U)\) which does not contain 1. The right side of (10) is therefore bounded below by some positive constant, and so (10) implies that 1 is not in \(S(f)\). Hence \(S(f) \neq H^2(U^2)\).

**Reference**


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